New analysis of the Klein-Gordon equation within the theory of deformed cross products. Introduction.

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The Klein-Gordon equation describing the propagation of massive waves is currently analyzed with Dirac's ideas and works. His equation and his matrices are strategic tools for everyone studying the standard model for particles. The initial goal of this document is to propose an alternative analysis for that crucial equation and to explore the consequences of this proposition. Within that approach, the Euclidean geometry of our everyday world reveal unexpected visages; for example, isotropic wave vectors and positions must be introduced into the debate. This document suggests a mechanism for the discovery of masses carried by these waves and initiates a procedure for the quantification of the area metrics.

Key words : Klein-Gordon equation, Deformed cross products,masses,neutrinos.

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1 The context and the challenge.

1.1 The Smale's problem number 9 and the search for a plausible theory of quantum gravity.

At the very beginning of the 20^{th} century, classifications have been proposed for physical (Hilbert - H) and for mathematical (Smale - S) unsolved questions [[01]].

In general, pure theoretical speculations don't attract so much attention in the public and at best turn in closed specialized circles. The immediate conclusion is that one certainly increases the audience in focusing on a well referred problematic (a H- or S- problem). A second strategic choice going in that direction (more readers) is made in writing in a language that most people read, understand, and sometimes speak; the Chinese people as well.

My central topic in mathematics is the elaboration of a theory: (i) deforming tensor products involving two elements in a vector space E(D, K) with D dimensions and a background K; (ii) explaining then how and why the obtained deformations can be decomposed in M(D, K) × E(D, K). My quest in physics has to do with the formulation of a plausible theory of quantum gravity. The mathematical part can indirectly be related to the S-9 problem [[01](a): pp. 11-12; [01](b); [01](c); §2, p.6]. The other part is a new episode in a old and long story paved with miles stones like S. Carlip's pioneer work [[02]] or, more recently, with panoramic overviews like in [[03]].

1.2 Representing particles: a strategic question.

As recalled in [[04]; §2, p. 3]: "(Citation) Much of the non-trivial physics (of scattering amplitudes) traces back to the simple question: "What is a particle?" (end of the citation)". And, actually, particles are though as irreducible representations of the Poincaré group and essentially described via their (kinetic) momentum, **p**.

1.3 An answer: analyzing the Klein-Gordon equation with the intrinsic method.

Remark 1.1. *Why the intrinsic method exploring the decomposition of deformed cross products can be involved?*

There are at least two important facts explaining why the mathematical method which has been developed in [[a]] can be involved when one studies the Klein-Gordon equation (K.G.E in this document):

1. **The formalism of the Klein-Gordon equation allows the introduction of a polynomial of degree two into the discussion.**
   The K.G.E in curved space-time has the formalism [[05]; p. 44, (3.26)]:

$$g^{\alpha\beta} \,.\, \frac{\partial^2 \phi(\mathbf{x})}{\partial x^\alpha x^\beta} \;+\; (m^2 \;+\; \xi \,.\, R(\mathbf{x}))\,.\,\phi(\mathbf{x}) \;=\; 0$$

   I shall start with its formulation in a space-time without curvature (R = 0) in re-introducing the universal constant c and $\hbar$ as in [[05]; p. 4, (6)]:

$$g^{\alpha\beta} \,.\, \frac{\partial^2 \phi({}^{(4)}\mathbf{x})}{\partial x^\alpha x^\beta} \;+\; \frac{m^2 \,.\, c^2}{\hbar^2} \,.\, \phi({}^{(4)}\mathbf{x}) \;=\; 0 \tag{1}$$

   The elementary solutions for the free fields version of that equation concern particles with an energy E and a kinetic momentum **p** and they are well-known [[06]; p. 16, (43)]:

$$\phi({}^{(4)}\mathbf{x}) \;=\; \phi_0({}^{(4)}\mathbf{x}) \,.\, exp^{\frac{i}{\hbar} \,.\, ({}^{(3)}\mathbf{p} \,.\, {}^{(3)}\mathbf{x} \,-\, E \,.\, t)} \tag{2}$$

   where ϕ_0 is a constant spinor; basic knowledge concerning the theory of spinors can be discovered in [[07]].

Remark 1.2. *Preliminaries.*

Let recall:

- The dispersion relation of light propagating in vacuum [[06]; p. 4, (5)]:

$$E^2 = m^2 . c^4 + c^2 . <^{(3)} \mathbf{p}, {}^{(3)}\mathbf{p} >_{Id_3} \tag{3}$$

- Basic relations connecting the characteristics of a wave:

$$c = \lambda . \nu \, ; \; 2 . \pi . \nu = \omega ; \; 2 . \pi . c = \lambda . \omega$$

- The fundamental equivalences which have been introduced in quantum mechanics:

$$E = h . \nu = \hbar . \omega \, ; \; \mathbf{p}^* = \hbar . \mathbf{k}^* \, ; \; E_0 = m . c^2 \tag{4}$$

- The Equ.(2) contains:
  (a) an implicit and non-obvious information about the geometrical context in which the mathematical discussion takes place. Indeed, that equation can be rewritten as:

$$\phi(^{(4)}\mathbf{x}) = \phi_0(^{(4)}\mathbf{x}) . exp^{\frac{i}{\hbar} . <^{(4)}\mathbf{p}, \, ^{(4)}\mathbf{x}>_{[\eta]}} \tag{5}$$

  where the square and diagonal (4-4) matrix $[\eta]$ denotes the fact that the geometry is "Minkowski" with signature (+ - - -).
  (b) an uncertainty concerning the mathematical writing; I mean: "Are the vectors **x** and **p** written under their co-variant or under their contra-variant formulation?" The fact that $[\eta] = [\eta]^{-1}$ does not facilitate the research of a well-motivated answer.

  Let discuss these topics more deeply:
  (a) In a discussion concerning only flat universes (my choice R = 0), the Petrov's classification [[08]] should be recalled. There is no obligation to reduce the discussion to a geometry which is Minkowski; the geometry can be represented by any element [G] in M(4, R). A. Einstein did make the choice to work with a symmetric metric in his seminal work [[09]]. This will also be mine in that document:

$$[G] = [G]^t \in M(4, R) \tag{6}$$

  (b) Since (i) the metric is a symmetric tensor and (ii) the vector space E(4, R) is isomorphic to its dual space E*(4, R):

$$g_{\alpha\beta} . p^{\alpha} . x^{\beta}$$
$$=$$
$$g_{\alpha\beta} . (g^{\alpha\chi} . p_{\chi}) . (g^{\beta\delta} . x_{\delta})$$
$$=$$
$$g_{\alpha\beta} . (g^{\alpha\chi} . g^{\beta\delta} . p_{\chi} . x_{\delta})$$
$$=$$
$$(g^{\chi\alpha} . g_{\alpha\beta} . g^{\beta\delta}) . p_{\chi} . x_{\delta}$$
$$=$$
$$g^{\chi\delta} . p_{\chi} . x_{\delta}$$

  Short:

$$<^{(4)} \mathbf{p}, \, ^{(4)}\mathbf{x} >_{[G]} = <^{(4)} \mathbf{p}^*, \, ^{(4)}\mathbf{x}^* >_{[G]^{-1}} \in R \tag{7}$$

Remark 1.3. *Convention.*

In my work, to avoid future misinterpretations, I adopt the following convention. In harmony with the usual mathematical notation but in opposition with the notation in physics, the contra-variant version of a vector is denoted with a bold letter; its co-variant version too but the co-variance will be signed by an asterisk on the right side of this letter.

With these conventions, the elementary solutions for the K.G.E in the flat regions of our universe can be written:

$$\phi({}^{(4)}\mathbf{x}) \tag{8}$$
$$=$$
$$\phi_0({}^{(4)}\mathbf{x}) \,.\, exp^{\frac{i}{\hbar}\,.\,<{}^{(4)}\mathbf{p}\,.\,{}^{(4)}\mathbf{x}>_{[G]}}$$
$$=$$
$$\phi_0({}^{(4)}\mathbf{x}) \,.\, exp^{\frac{i}{\hbar}\,.\,<{}^{(4)}\mathbf{p}^*\,.\,{}^{(4)}\mathbf{x}>_{Id_4}}$$

Remark 1.4. *Successive partial derivations.*

Since:

$$<^{(4)} \mathbf{p}^* \,.\, {}^{(4)}\mathbf{x} >_{Id_4} = p_\alpha \,.\, x^\alpha$$

It is easy to get:

$$\frac{\partial \phi({}^{(4)}\mathbf{x})}{\partial x^\beta} \;=\; \frac{i}{\hbar} \,.\, (p_\beta \,+\, \frac{\partial p_\alpha}{\partial x^\beta} \,.\, x^\alpha) \,.\, \phi({}^{(4)}\mathbf{x}) \tag{9}$$

And, in a second step:

$$\frac{\partial^2 \phi({}^{(4)}\mathbf{x})}{\partial x^\chi \partial x^\beta} \tag{10}$$
$$=$$
$$\frac{i}{\hbar} \,.\, \frac{\partial (p_\beta \,+\, \frac{\partial p_\alpha}{\partial x^\beta} \,.\, x^\alpha)}{\partial x^\chi} \,.\, \phi({}^{(4)}\mathbf{x}) \,+\, \frac{i}{\hbar} \,.\, (p_\beta \,+\, \frac{\partial p_\alpha}{\partial x^\beta} \,.\, x^\alpha) \,.\, \frac{\partial \phi({}^{(4)}\mathbf{x})}{\partial x^\chi}$$
$$=$$
$$\{\frac{i}{\hbar} \,.\, (\frac{\partial p_\beta}{\partial x^\chi} + \frac{\partial p_\chi}{\partial x^\beta} + \frac{\partial^2 p_\alpha}{\partial x^\chi \partial x^\beta} \,.\, x^\alpha) - \frac{1}{\hbar^2} \,.\, (p_\beta + \frac{\partial p_\alpha}{\partial x^\beta} \,.\, x^\alpha) \,.\, (p_\chi + \frac{\partial p_\epsilon}{\partial x^\chi} \,.\, x^\epsilon)\} \,.\, \phi({}^{(4)}\mathbf{x})$$

Remark 1.5. *New formalism for the K.G.E.*

Let now inject the Equ.(10) into the K.G.E, Equ.(1):

$$g^{\chi\beta} \,.\, \frac{\partial^2 \phi({}^{(4)}\mathbf{x})}{\partial x^\chi \partial x^\beta} \;+\; \frac{m^2 \,.\, c^2}{\hbar^2} \,.\, \phi({}^{(4)}\mathbf{x}) \tag{11}$$
$$=$$
$$\{g^{\chi\beta} \,.\, \{\frac{i}{\hbar} \,.\, (\frac{\partial p_\beta}{\partial x^\chi} + \frac{\partial p_\chi}{\partial x^\beta} + \frac{\partial^2 p_\alpha}{\partial x^\chi \partial x^\beta} \,.\, x^\alpha) - \frac{1}{\hbar^2} \,.\, (p_\beta + \frac{\partial p_\alpha}{\partial x^\beta} \,.\, x^\alpha) \,.\, (p_\chi + \frac{\partial p_\epsilon}{\partial x^\chi} \,.\, x^\epsilon)\} + \frac{m^2 \,.\, c^2}{\hbar^2}\}$$
$$\times$$
$$\phi({}^{(4)}\mathbf{x})$$
$$=$$
$$0$$

In a free-fall motion (geodesic), there is no force acting on the light and this expression is drastically simplified:

$$\frac{d\mathbf{p}}{ds} = \mathbf{0}$$

$$\Downarrow$$

$$\{g^{\chi\beta} \, . \, \frac{\partial^2 \phi({}^{(4)}\mathbf{x})}{\partial x^\chi \partial x^\beta} + \frac{m^2 \, . \, c^2}{\hbar^2}\} \, . \, \phi({}^{(4)}\mathbf{x}) \tag{12}$$

$$=$$

$$-\frac{1}{\hbar^2} \, . \, \{g^{\chi\beta} \, . \, p_\chi \, . \, p_\beta - m^2 \, . \, c^2\} \, . \, \phi({}^{(4)}\mathbf{x})$$

$$= (seeEqu.(4-b)$$

$$-\{g^{\chi\beta} \, . \, k_\chi \, . \, k_\beta - \frac{m^2 \, . \, c^2}{\hbar^2}\} \, . \, \phi({}^{(4)}\mathbf{x})$$

$$=$$

$$0$$

This injection reveals the existence of a polynomial of degree two, Λ, depending on the co-variant components of the wave vector $\mathbf{k}^*$:

$$\{g^{\chi\beta} \, . \, \frac{\partial^2 \phi({}^{(4)}\mathbf{x})}{\partial x^\chi \partial x^\beta} + \frac{m^2 \, . \, c^2}{\hbar^2}\} \, . \, \phi({}^{(4)}\mathbf{x}) = \Lambda({}^{(4)}[G], \, {}^{(4)}\mathbf{k}^*) \, . \, \phi({}^{(4)}\mathbf{x}) = 0$$

with:

$$\Lambda({}^{(4)}[G]^{-1}, \, {}^{(4)}\mathbf{k}^*) = g^{\chi\beta} \, . \, k_\chi \, . \, k_\beta - \frac{m^2 \, . \, c^2}{\hbar^2} \tag{13}$$

The solutions of that polynomial are those of the K.G.E. The Equ.(7) gives the possibility to prefer that other formulation:

$$\Lambda({}^{(4)}[G], \, {}^{(4)}\mathbf{k}) = g_{\chi\beta} \, . \, k^\chi \, . \, k^\beta - \frac{m^2 \, . \, c^2}{\hbar^2} \tag{14}$$

Example 1.1. *When the metric is Minkowski.*

If the metric is Minkowski:

$$\phi({}^{(4)}\mathbf{x}) = \phi_0(\mathbf{x}) \, . \, exp^{i \, . \, ({}^{(3)}\mathbf{k}^* \, . \, {}^{(3)}\mathbf{x} - \omega \, . \, t)}$$

Successive partial derivations by respect for the components of the event ${}^{(4)}\mathbf{x}$ bring a set of relations:

$$\frac{\partial \phi({}^{(3)}\mathbf{x}, \, t)}{\partial x^a} = i \, . \, k_a \, . \, \phi({}^{(3)}\mathbf{x}, \, t) \, ; \; \frac{\partial \phi({}^{(3)}\mathbf{x}, \, t)}{\partial x^0} = -i \, . \, \omega \, . \, \phi({}^{(3)}\mathbf{x}, \, t) \tag{15}$$

and:

$$\frac{\partial^2 \phi({}^{(3)}\mathbf{x}, \, t)}{\partial x^a \partial x^b} = -\, k_a \, . \, k_b \, . \, \phi({}^{(3)}\mathbf{x}, \, t) \tag{16}$$

$$\frac{\partial^2 \phi({}^{(3)}\mathbf{x}, \, t)}{\partial x^a \partial x^0} = \omega \, . \, k_a \, . \, \phi({}^{(3)}\mathbf{x}, \, t)$$

$$\frac{\partial^2 \phi({}^{(3)}\mathbf{x}, \, t)}{\partial^2 x^0} = -\, \omega^2 \, . \, \phi({}^{(3)}\mathbf{x}, \, t)$$

In injecting these relations into the K.G.E, I get:

$$\{-g^{ab} \, . \, k_a \, . \, k_b + (g^{a0} + g^{0a}) \, . \, \omega \, . \, k_a - g^{00} \, . \, \omega^2 + \frac{m^2 \, . \, c^2}{\hbar^2}\} \, . \, \phi({}^{(3)}\mathbf{x}, \, t) = 0$$

The maneuver reveals the existence of a polynomial of degree two depending on the co-variant components of the spatial wave vector $\mathbf{k}^*$:

$$\forall\, \phi({}^{(3)}\mathbf{x},\, t) :$$

$$\exists\, \Lambda({}^{(4)}[G],\, {}^{(4)}\mathbf{k}^*) \;=\; g^{ab} \,.\, k_a \,.\, k_b - (g^{a0} + g^{0a}) \,.\, \omega \,.\, k_a + g^{00} \,.\, \omega^2 - \frac{m^2 \,.\, c^2}{\hbar^2} \tag{17}$$

Remark 1.6. *A 3 + 1 decomposition.*

Let consider the Equ.(14) and state that it can always be decomposed in a 3 + 1 manner as:

$$\Lambda({}^{(4)}[G],\, {}^{(4)}\mathbf{k}) \;=\; g_{ab} \,.\, k^a \,.\, k^b \;+\; [(g_{a0} \;+\; g_{0a}) \,.\, k^0] \,.\, k^a \;+\; [g_{00} \,.\, (k^0)^2 \;-\; \frac{m^2 \,.\, c^2}{\hbar^2}] \tag{18}$$

Hence, it can be interpreted as a polynomial of degree two depending on the three spatial contra-variant components of the wave vector. This fact is justifying the second argument of this demonstration.

2. **In a three-dimensional space, each polynomial of degree three can be interpreted as the signature of the presence of some deformed cross product that has been non-trivially decomposed.**
   Let consider a deformed cross product [1] and let suppose that there exists at least one non-trivial decomposition corresponding to it. Then, due to the initial theorem [[a]; theorem 3.1, p. 10], there is a polynomial of degree two depending on the components of the projectile (synonym: the first argument in the product at hand).

$$\exists\, ([P],\, \mathbf{z}) \;:\; |[\mathbf{q}_1, \mathbf{q}_2]_{[A]} >= [P] \,.\, |\mathbf{q}_2 > + |\mathbf{z} > \tag{19}$$

$$\Downarrow$$

$$\exists\, \Lambda(\mathbf{q}_1) \;=\; |[P] \;-\; {}_{[A]}\Phi(\mathbf{q}_1)| \;=\; d_{ab}\, q_1^a \,.\, q_1^b \;+\; d_a \,.\, q_1^a \;-\; |P|$$

The formal similitude between the equations (18) and (19) is obvious and obtained through the four simple relations (recall that, in this theory, the metric is symmetric):

$$\mathbf{q}_1 \;=\; \mathbf{k} \tag{20}$$

$$[D_{ab}] \;=\; [G]$$

$$\mathbf{d}^* \;=\; 2 \,.\, k^0 \,.\, \mathbf{g}^*;\; \mathbf{g}^* \,:\, (g_{01},\, g_{02},\, g_{03})$$

$$-|P| \;=\; g_{00} \,.\, (k^0)^2 \;-\; \frac{m^2 \,.\, c^2}{\hbar^2}$$

This fact justifies the:

Lemma 1.1. *The intrinsic method which has been presented in [[a]] allows at least a formal analysis of the Klein-Gordon equation (alias: K.G.E in this document) for free fields and for fields stemming from a potential proportional to the wave function* [2]*).*

[1] In a three-dimensional vector space (D = 3), the deformation is due to a (3-3) matrix [A], an element in M(3, K). In this document, K = C, the set of complex numbers.

[2] These situations are resulting in a re-scaling of the mass. Otherwise, they receive a totally similar mathematical treatment than the free fields cases.

The theory studying the deformed cross products suggests that the elementary solutions of the Klein-Gordon equation are related to a set of deformed cross products.

$$\exists\,\Lambda({}^{(4)}[G],\,{}^{(4)}\mathbf{k}^*) \;=\; g_{\chi\beta}\,.\,k^{\chi}\,.\,k^{\beta} \;-\; \frac{m^2\,.\,c^2}{\hbar^2}$$

$$\Downarrow$$

$$\exists\,([P],\,\mathbf{z})\;:\;|[\mathbf{k},\mathbf{q}_2]_{[A]}> = [P]\,.\,|\mathbf{q}_2> + |\mathbf{z}>$$

This point will be developed further in the next sections.

2 Acknowledgments.

Because I am an independent researcher, I am restricting myself to work with books related to the foundations of mathematics or of physics and to academic documents. Sometimes, I am obliged to loan some elements in the arXiv library. I warmly acknowledge the authors ... and the patience of my wife.

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