© Thierry PERIAT: C*algebras for spaces equipped with a deformed tensor product, the question of the norm and the extrinsic method,ISBN 978-2-36923-144-8,EAN 9782369231448, v1, 15 September 2020.

This document is a relooked version of an exploration which I started and first published in 2008 under the comic title: BS for ET, in extenso: Banach spaces for the theory of the (E) question. It now has a French matriculation.

With the gravitational term appearing in the covariant version of the Lorentz law in mind, the main topic already was the construction of a C*algebra structure for spaces equipped with a deformed tensor product.

Some technical difficulties concerning the norm was forcing me to look for new roads. My so-called "extrinsic method" found here its first application and came to the rescue; a long time before I got the inspiration for a link with E. B. Christoffel' work.

1. THE PROPOSED SEMI-NORM IS NOT THE WORST.

1.1. REMARK: MOTIVATION

In previous works we began the construction of a C*-algebra and encountered a difficulty with the expression proposed for the norm. Indeed, we discovered that it only was a semi-norm. Furthermore, some of the necessary conditions to build this C*-algebra has imposed the necessity to work with an "Iso" subset (in extenso: with a set of isotropic elements; $\otimes_A(\mathbf{a}, \mathbf{a}) = 0$, whilst neither $\mathbf{a}$ nor A vanish) and any element of such set has a vanishing (semi)-norm. So that we are in a "cul-de-sac" with our initial proposition; and that we must now explore some other road to progress.

1.2. REMARK

One "generic" condition for a given expression to be a norm is $||\, \lambda.\, \mathbf{A}|| = |\lambda|.\, ||\mathbf{A}||$; see [01; page 1; definition 1.1 (c)]. Until now, our proposition to build a norm was $||\, \mathbf{A}|| = |\, \mathbf{A}.\, \otimes_A(\mathbf{A}, \mathbf{A})|$ where the symbol "| … |" is the traditional norm on K = $\mathbb{R}$ or $\mathbb{C}$, the symbol "**.**" is the scalar product defined on (E_N, K) and the symbol "⌂$_A$(**…**, **…**)" is the extended product also defined on (E_N, K) with the cube A.

Although offering a natural link with the matrix [J] defining the classical cross product in a three-dimensional Euclidean space (see the dissertation on the intrinsic method), the immediate consequence of this proposition on the generic condition is in fact a restriction concerning the subset of K where the numbers λ can be chosen; here the restriction writes: $|\lambda^3| = |\lambda|$ and this is resulting in $\lambda \in \{-1, -j, -j^2, 1, j, j^2\} \otimes \{z \in K \mid |z| = 1\}$.

1.3. DEFINITION

Let us suppose we would have defined the expression:

$$f(\mathbf{A}) = |\mathbf{A}.\, \{\mathbf{A} + \otimes_A(\mathbf{A}, \mathbf{A}) + \otimes_A(\otimes_A(\mathbf{A}, \mathbf{A}), \mathbf{A}) + \otimes_A(\otimes_A(\otimes_A(\mathbf{A}, \mathbf{A}), \mathbf{A}), \mathbf{A}) + \ldots\}|$$

Let us calculate f(λ. **A**). We find:

$$f(\lambda.\, \mathbf{A}) = |\lambda^2.\, \mathbf{A}^2 + \lambda^3.\, \mathbf{A}.\, \otimes_A(\mathbf{A}, \mathbf{A}) + \lambda^4.\, \mathbf{A}.\, \otimes_A(\otimes_A(\mathbf{A}, \mathbf{A}), \mathbf{A}) + \ldots\, |.$$

1.4. REMARK

Let us take the vector **A** into the set of the idempotents relatively to the extended product; namely:

$$\text{Idem}\ (E_N, K, \otimes_A) = \{\mathbf{A} \in (E_N, K) \mid \otimes_A(\mathbf{A}, \mathbf{A}) = \mathbf{A}\}$$

It follows:

$$f(\lambda.\,\mathbf{A}) = |\lambda^2 + \lambda^3 + \lambda^4 + \ldots + \lambda^{p-1}|.\ |\mathbf{A}^2|$$

In the same condition ($\mathbf{A} \in$ Idem $(E_N, K, \otimes_A)$), $f(\mathbf{A}) = (p-2).\ |\mathbf{A}^2|$; where necessarily $p \in \mathbb{N}^* - \{1\}$. Independently of the element **A**, an evident way to get the necessary generic condition is then:

$$|\lambda|.\,(p-2) = |\lambda^2 + \lambda^3 + \lambda^4 + \ldots + \lambda^{p-1}|.$$

1.5. REMARK

If λ is one of the p roots of 1 in $\mathbb{C}$, then the above condition is reduced to:

$$(p-2).\ |\lambda| = (p-2) = |1 + \lambda|$$

because, in that case, $|\lambda| = 1$. Let us translate it in $\mathbb{R} \times \mathbb{R}$. We get:

$$(p-2)^2 = |1 + \lambda|^2$$

$$(p-2)^2 = (1 + x)^2 + y^2 = 2.\,(1 + x) \text{ with } x = \cos(2\pi\theta/p),\ y = \sin(2\pi\theta/p) \text{ and } \theta = 0, 1, \ldots, p-1.$$

Let us develop this equation and verify that:

$$x = ½.\,(p-2)^2 - 1 = \cos(2\pi\theta/p)$$

To clarify this problematic, let us try $p = 3$ (which is in fact a case remarkably like the case develop in § 1.2). We have to verify if the following equation makes sense: $-½ = \cos(2\pi\theta/3)$ for $\theta = 0, 1, 2$. The answer is: yes for $\theta = 1$ and 2. Let us try $p = 4$; we have to verify if the following equation makes sense: $1 = \cos(\pi\theta/2)$ for $\theta = 0, 1, 2, 3$. The answer is: yes for $\theta = 0$ only. But note that in that case $\lambda = 1$, only. Let us try $p = 5$; we must verify if the above equation makes sense. The answer is immediately: no; because we would have a cosine greater than 1; which is impossible.

1.6. CONCLUSION

Apparently, since a function $f(\mathbf{A}) = |\mathbf{A}^2|$ is the traditional definition for a norm and consequently represents no progress, the only new and meaningful proposition to extend the notion of norm with the help of the deformed tensor product seems to be the function $f(\mathbf{A}) = |\mathbf{A}.\ \otimes_A(\mathbf{A}, \mathbf{A})|$ proposed in § 1.2.

Unfortunately, even if it is coinciding with the traditional norm for any element of the Idem $(E_N, K, \triangle_A)$ set, it is unsatisfactory because of the restriction on λ when **A** is any element of (E_N, K).

1.7. REMARK

The real number $f(\mathbf{A})$ is the measure of the projection on the axis represented by the vector **A** of the extended square product of this vector. It vanishes if this square product is orthogonal to the initial vector.

1.8. REMARK

The real number $f(\mathbf{A})$ also is what we did call "the equivalent (or associated) scalar" associated with the square product $\otimes_A(\mathbf{A}, \mathbf{A})$. Thus, we should perhaps remember the investigation made with the purpose to compare it with the scalar obtained when this deformed tensor product is decomposed (the extrinsic method).

2. PRECISIONS

Our claim is the construction of C*-algebras on a space vector (E_N, ℂ) equipped with a deformed tensor product $\otimes_A$. Here, we now focus our self on algebras "for topologists" [02; §2; page 3]. We thus must first build involutive and Banach algebras.

2.1. DEFINITION: INVOLUTIVE ALGEBRA.

Five properties characterize this type of algebra:

a. The deformed tensor product must be **associative**. Recall: as already demonstrated in former sections, the existence of an associative deformed tensor product automatically defines a multiplicative morphism $A\Phi$ between {(E_N, ℂ), $\otimes_A$} and {M_N(ℂ), .}.

b. **Involution** itself. There exists a map {(E_N, ℂ), $\otimes_A$} → {(E_N, ℂ), $\otimes_A$}: **a** → **a*** such that **a** = a**. This formal definition is a door opening a room with many possibilities. But the choice that will be done here will have important consequences on the other imposed conditions. We shall illustrate this remark below.

c. (**a** + **b**)* = **a*** + **b***. Let us suppose that we define the above map (adjoint element) with **a*** = $\otimes_A$(**a**, **a**). That condition c) imposes an anticommutative deformed tensor product when, furthermore, the elements **a** are isotropic vectors. If we make another choice: **a* =** $\otimes_A$ (**a**, $\otimes_A$(**a**, **a**)), the condition c) is now in general: $\otimes_A$(**a + b**, $\otimes_A$(**a + b**, **a + b**)) = $\otimes_A$ (**a**, $\otimes_A$(**a**, **a**)) + $\otimes_A$(**b**, $\otimes_A$(**b**, **b**)); this is imposing another context and this is implying a totally different condition.

d. (λ. **a**)* = λ^*. **a***, $\forall\lambda \in$ ℂ where λ^* is the ordinary conjugate complex number of λ. Let us suppose that the adjoint element is **a*** = $\otimes_A$(**a**, **a**). The condition d) imposes $\lambda^2 = \lambda^*$. If we now make the choice: **a*** = $\otimes_A$(**a**, $\otimes_A$(**a**, **a**)), the condition d) is now: $\lambda^3 = \lambda^*$.

e. $\otimes_A$(**a**, **b**)* = $\otimes_A$(**b***, **a***). Once more time the effective consequence of that condition depends on the choice made for the definition of the involution.

2.2. REMARK

Let suppose that $\lambda = i \in$ ℂ | ($i^2 + 1 = 0$), then $\lambda^3 = i^3 = -i = \lambda^* = -\lambda$. This relation characterizes a set of matrices in M_4(ℂ) used to write the world line Green functions [04; page 5; (2.4)]. This fact motivates our interest for unital Banach algebras in which every non zero element is invertible because such algebras are isomorphic to ℂ by the Gelfand Mazur theorem [02; page 8; theorem 2.19]. Each of the relations $\lambda^2 = \lambda^*$ and $\lambda^3 = \lambda^*$, is a direct consequence of the definition made for the adjoint element.

The general result of that way of doing is the determination of a subset, said $\mathcal{C} \subseteq$ ℂ, for which the d) condition holds. The existence of an isomorphism between ℂ and a given unital Banach algebra in which every nonzero element is invertible implies those of a representation of any subset of ℂ in that Banach algebra. In that case, $\mathcal{C}$ owns a representation too. For the pragmatic application at hand [04], the powers 0, 1 and 2 of the matrix $\mathcal{Z}$ and of its adjoint are expected to be some elements of this representation. This way of thinking is interesting only if the isomorphism preserves the relation (valid in ℂ) resulting from the respect of the condition d) for a given choice of the adjoint. The interesting point here will be in fact to explore the transformed of $\mathcal{C} \otimes$ {(E_N, ℂ), $\otimes_A$} via an isomorphism and to verify if we can with it, for example, write the elements in the subset of M_4(ℂ) generated by the powers 0, 1 and 2 of the matrix $\mathcal{Z}$.

2.3. REMARK

$M_N(\mathbb{C})$ is a C*-algebraic structure for the operator norm [02; page 4; example 2.3].

2.4. REMARK

The initial strategy is (i) the definition of a sub-algebra of $M_N(\mathbb{C})$ equipped with a C*-algebraic structure and (ii) the precision of the conditions for which the multiplicative morphism is a bijection.

The surjection of the multiplicative morphism $_A\Phi$ has already been explored. The injection remains an open question:

$$? : {}_A\Phi(\mathbf{a}) = {}_A\Phi(\mathbf{b}) \to \mathbf{a} = \mathbf{b}. \text{ Per definition: } {}_A\Phi(\mathbf{a}) = [A^{\alpha}{}_{\gamma\beta}.\ a^{\gamma}].$$

2.5. REMARK

When the multiplicative morphism $_A\Phi$ is a bijection, we can envisage the definition of the resolvent.

2. 6. DEFINITION: RESOLVENT

Let us suppose that range($_A\Phi$) is a C*-algebra. The resolvent of a matrix [M] is the set of all matrices [Z] in $M_N(\mathbb{C})$ such that {[M] – [Z]}-1 is also in range($_A\Phi$). Since $M_N(\mathbb{C})$ is a C*-algebraic structure for the operator norm, [Z] has a norm. The spectrum of [M] can be defined as being the biggest norm that can be calculated for an element in its resolvent. And if $_A\Phi$ is a bijection, the spectrum of the source of [M] can also be defined. It is the biggest norm that can be calculated for an element in the resolvent of the source of [M].

2.7. REMARK: THE NORM.

Accordingly to the considerations made by the Pr. Connes [05; page13], there is in fact no liberty for the choice of the norm. It should absolutely be (spectrum of $\otimes_A(\mathbf{a}^*, \mathbf{a}))^{1/2}$. For the adjoint element $\mathbf{a}^* = \otimes_A(\mathbf{a}, \mathbf{a})$ it is: (spectrum of $\otimes_A(\otimes_A(\mathbf{a}, \mathbf{a}), \mathbf{a}))^{1/2}$. Note that it is not so far from our intuitive proposition which, unfortunately, only was a semi-norm. With this strategy, we should now be able to build a Banach space for the Theory of the (E) question.

3. LOOKING FOR SOMETHING NEW.

3.1. DEFINITION

Consider a new proposition:

$$g(\mathbf{A}) = |\mathbf{A}.\ \{\mathbf{A} + ½.\ ⌂▾_A(\mathbf{A}, \mathbf{A}) + 1/6.\ ⌂▾_A(⌂▾_A(\mathbf{A}, \mathbf{A}), \mathbf{A}) + 1/24.\ ⌂▾_A(⌂▾_A(⌂▾_A(\mathbf{A}, \mathbf{A}), \mathbf{A}), \mathbf{A}) + \ldots\}|$$

3.2. REMARK

Let us suppose that the extended product is defined with the help of a cube A allowing to interpret any $⌂_A(\mathbf{A}, \mathbf{B})$ as being equivalent to a derivation $\partial_{\mathbf{A}}\mathbf{B}$. Then the function f(**A**) in § 1.3 can be written:

$$g(\mathbf{A}) = |\mathbf{A}. \{\mathbf{A} + ½. \partial_{\mathbf{A}}\mathbf{A} + 1/6. \partial_{\mathbf{A}}\partial_{\mathbf{A}}\mathbf{A} + \ldots + [1/(p-1)!]. \partial_{\mathbf{A}}\ldots\partial_{\mathbf{A}}\mathbf{A}\}|$$

This is strongly remembering a Taylor's development; except that we are now working with vectors.

3.3. REMARK

Let us suppose that the extended product defined with the help of a cube A is associative and owns trivial splits. Then the function g(**A**) can be written:

$$g(\mathbf{A}) = |\mathbf{A}. \{\mathbf{A} + ½. \triangledown_A(\mathbf{A}, \mathbf{A}) + 1/6. \triangledown_A(\mathbf{A}, \triangledown_A(\mathbf{A}, \mathbf{A})) + \ldots\}|$$

$$g(\mathbf{A}) = |\mathbf{A}. \{\mathbf{A} + ½. {}_A\Phi(\mathbf{A}). \mathbf{A} + 1/6. {}_A\Phi^2(\mathbf{A}). \mathbf{A} + \ldots\}|$$

$$g(\mathbf{A}) = |\mathbf{A}. \{(I + ½. {}_A\Phi(\mathbf{A}) + 1/6. {}_A\Phi^2(\mathbf{A}) + \ldots). \mathbf{A}|$$

This is strongly remembering a Taylor's development and suggesting that:

$$g(\mathbf{A}) = |\mathbf{A}. \exp\{{}_A\Phi(\mathbf{A})\}. \mathbf{A}|$$

This is encouraging us to find circumstances for which the "norm" of the matrix $\exp\{{}_A\Phi(\mathbf{A})\}$ is 1.

4. CONVERGENCE

4.1. REMARK: THE "EQUIVALENT SCALAR" (AND THE METHOD ASSOCIATED WITH)

This procedure has still been developed extensively in diverse documents. Its essence is a comparison between a scalar obtained with the help of the split of a deformed tensor product (anyone; so inclusively a square one) and a scalar obtained as a Taylor' development. The comparison is possible if three conditions are realized simultaneously. The first of them implies proportionality between the two scalars. So that if one is vanishing, then the other also does. Since one of these scalars is built with the expression proposed for the norm, this is a fundamental remark.

The claim is to precise conditions for which the semi-norm vanishes. The hope is to discover circumstances for which a vanishing semi-norm is automatically associated with the vanishing of the vector **A**. In that circumstance the semi-norm becomes a norm.

The ad hoc "associated scalar" is related to $T = < d\mathbf{x}|\ \text{Hess}\ (h(\mathbf{x}))|\ d\mathbf{x} >$. Obviously, the condition $d\mathbf{x} = \mathbf{0}$ implies T = 0. Unfortunately, the converse is not automatically true and only implies: $[\partial^2_{\mu\nu}h(\mathbf{x}) - \Gamma_{\mu\nu}{}^{\theta}. \partial_\theta\, h(\mathbf{x})]. dx^\mu. dx^\nu = 0$.

The most important item becomes the study of the convergence of the proposed semi-norm with the so-called "associated scalar" obtained in supposing the existence of at least one non-trivial decomposition for the square deformed tensor product under consideration.

The idea is now to explore if the projection of the decomposition of a deformed tensor product is a limit for the proposed semi-norm. This is: we consider à priori the existence of a sequence in M_N (K) x (E_N, K, $\otimes_A$) $\{([P_\alpha], \mathbf{z}_\alpha)$ for $\alpha = 1, 2\ldots p\}$ and we try to discover circumstances for which the following proposition becomes true:

$$\text{Lim}_{\alpha\to\infty} |\mathbf{A}. \otimes_A (\mathbf{A}, \mathbf{A}) - \mathbf{A}. \{[P_\alpha]. \mathbf{A} + \mathbf{z}_\alpha\}| = 0 \text{ (convergence of type 1)}$$

Or, we consider à priori the existence of a sequence of cubes $A(\alpha)$ for α = 1, 2... p} and we try to discover circumstances for which the following proposition becomes true:

$$\mathrm{Lim}_{\alpha\to\infty} |\mathbf{A}. \otimes_{A(\alpha)} (\mathbf{A}, \mathbf{A}) - \mathbf{A}. \{[P]. \mathbf{A} + \mathbf{z}\}| = 0 \text{ (convergence of type 2)}$$

In both cases a convergence exists which is realizing a kind of harmony between a cube, a decomposition and the expression proposed for the norm. In the first case (type 1) a sequence of decompositions is converging so that one can hope to reach the value f(**A**) proposed to be a norm on (E_N, K, $\otimes_A$). In the other case (type 2) a sequence of deformed tensor products is converging so that one of them coincides with a given decomposition.

In the document exposing the extrinsic method we could propose a procedure to minimize any scalar s = **B.** $\otimes_A$(**A**, **B**) – **B.** {[P]. **B** + **z**} when **B** = d**A** and where "d" denotes the ordinary derivation[1] on (E_N, K). That is, given a deformed tensor product $\otimes_A$ (**A**, **B**) and an invertible metric, we can always find an element ([P], **z**) of M_N (K) x (E_N, K, $\otimes_A$) minimizing "s" up to the terms of the third order. Unfortunately, even if this is making the decomposition ([P], **z**) quasi-perfect, it is giving no absolute insurance for the existence of the decomposition. The proposed decomposition always is an approximation.

Since the scalar s is bilinear in **B**, the method can be extended without any change to cases where **B** = ± d**A**. It may be intuitively extended to cases where **B** = ± [X]. d**A** provided an ad hoc condition on the [X] matrices exist. So that an application to the problematic of the semi-norm can be accepted for cases where, also actually intuitively, **A** = ± [X]. d**A**; this is offering an interesting opportunity to study exponentially increasing or decreasing vectors. Quite more interesting at a theoretical level, this is giving us a natural opportunity to try to involve the "Dirac's Ansatz" [03; § 3.3; page 102) inside of our mathematical theory when **A** represents a spinor (a set of four wave functions if N = 4).

All these thoughts have especially important implications on our ability to analyze the Lorentz-Einstein Law (LEL) which is a special illustration when N = 4 and **A** represents the 4D speed vector of a particle. But with a little imperfection: the validity of this analysis pre-supposes (or implicitly supposes) that the LEL is a law only approximately true; or is secretly including terms of the third order into the residual acceleration.

4.2. REMARK

If N = 1, which is the easiest configuration, the realization of one of the two possible convergences exposed in prior paragraph automatically yields a polynomial form of degree 3 depending on "a", the unique coordinate of **A**. The generic formalism of this form is:

$$|A.\, a^3 - p.\, a^2 + z| = 0$$

All vectors of which the coordinate satisfies this equation realize the above convergence. But since in a 1D space the expression f(**A**) is a trivial cubic form A. a^3, the question of the semi-norm in such space is meaningless if it is real. That means: the expression f(**A**) = |A. a^3| satisfies the condition of "separation": f(**A**) = 0 → **A = 0** when K = ℝ. Please also remark that in such case, if the coordinate has the unit of a length, then the proposed nom is a volume.

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[1] The ordinary derivation is not a differential.

4.3. REMARK

For any N $\in \mathbb{N}^*$, let us suppose that one of the convergences is realized (type 1 or type 2). This is transforming the question of the semi-norm. Let us consider the type 1 one. The question is:

$$\mathbf{A}.\ \{[P].\ \mathbf{A} + \mathbf{z}\} = 0 \ \rightarrow ?\ \mathbf{A} = \mathbf{0}$$

If the convergence is obtained for a trivial decomposition, the question is more precisely:

$$\mathbf{A}.\ \{{}_{A}\Phi(\mathbf{A}).\ \mathbf{A}\} = 0 \ \rightarrow ?\ \mathbf{A} = \mathbf{0}$$

The above relation $\mathbf{A}.\ \{{}_{A}\Phi(\mathbf{A}).\ \mathbf{A}\} = 0$ can be realized in three cases:

1°) **A = 0**, which is perfect but without any interest.

2°) **A** is orthogonal to ${}_{A}\Phi(\mathbf{A}).\ \mathbf{A}$; see later.

3°) ${}_{A}\Phi(\mathbf{A}).\ \mathbf{A} = \mathbf{0}$. Let us first examine the third case.

3°a) Either $|\ {}_{A}\Phi(\mathbf{A})\ | \neq 0$. Traditional rules (Kramer's system) imply that **A = 0**. But in that case, ${}_{A}\Phi(\mathbf{A}) = [0]$ and $|\ {}_{A}\Phi(\mathbf{A})| = 0$; this is a non-sense.

3°b) Or $|\ {}_{A}\Phi(\mathbf{A})| = 0$. There is no obligation for **A** to vanish.

For example, consider the special case N = 3 and $\blacktriangledown A = \blacktriangledown \varepsilon$. The expression under study always vanishes without any obligation for **A** to vanish:

$${}_{A}\Phi(\mathbf{A}).\ \mathbf{A} = \mathbf{A} \wedge \mathbf{A} = \mathbf{0}$$

4.4. REMARK

Another versus of the generic condition to get a norm inside of our own approach is [02; page 3; § 2.1] $\otimes_A(\lambda.\ \mathbf{A},\ \lambda.\ \mathbf{A}) = \lambda^*.\ \otimes_A(\mathbf{A},\ \mathbf{A})$. This yield: $\lambda^2 = \lambda^*$ (conjugate of). More precisely: $x^2 + y^2 = x$ and $2.\ x\ .\ y = -\ y$. We still had the opportunity to demonstrate that it implies: $\lambda \in \{1, j, j^2\}$. This is a more restricting result than those of § 1.2.

4.5. REMARK

If we try to introduce the notion of Cauchy sequences, we must develop it in different ways. First possibility, we consider the type 2 convergence defined in § 3.1 and we write:

4.6. DEFINITION

A sequence of cube is de facto defining a sequence of "semi-norms" for any given vector **A**. This sequence is Cauchy if:

$$\forall\ \varepsilon > 0, \exists\ N > 0, \forall\ \alpha, \beta \geq N,\ |\mathbf{A}.\ \triangle_{\blacktriangledown A(\alpha)}(\mathbf{A},\ \mathbf{A}) - \mathbf{A}.\ \triangle_{\blacktriangledown A(\beta)}(\mathbf{A},\ \mathbf{A})| < \varepsilon$$

Obviously, if a sequence of semi-norm for **A** is Cauchy it does not obligatory converges.

4.7. DEFINITION

A sequence of cube is de facto defining a sequence of "semi-norms" for any given vector **A**. This sequence is convergent if:

$$\forall \varepsilon > 0, \exists N > 0, \forall \alpha \geq N, |\mathbf{A}. \triangle\!\blacktriangledown_{A} (\mathbf{A}, \mathbf{A}) - \mathbf{A}. \triangle\!\blacktriangledown_{A(\alpha)} (\mathbf{A}, \mathbf{A})| < \varepsilon$$

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