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Abstract : The theory of the (E) question is concerned with the decomposition (division) of deformed tensor (resp. Lie) products. A first mathematical method (the intrinsic one) has been developed for the decomposition of deformed cross products. It only works in three-dimensional spaces and brings incomplete results.
This document proposes a second approach bringing complete results whatever the dimension of the mathematical space is, i.e. : the main and the residual parts of each decomposition. But, the method is plagued with a logical uncertainty.
Since the main part of the decomposition can be linked with the concept of fluctuation operator in mathematical physics, I explore further the domain of definition of that theory. I explain that both methods can be calibrated in any three-dimensional space and I realize the calibration. I also analyse the three-dimensional Euclidean limit in detail.
It turns out that in a three-dimensional context, that approach mainly concerns the deformations of cross products involving isotropic singular vectors related to diverse proper polynomials of degree two.

Key words : Mathematical methods; deformed tensor (resp. Lie) products; analysis.

1 Basics.

Definition 1.1. *Cube*

Per definition, in that theory, a cube is a mathematical object of which the components (i) are elements arbitrarily chosen in a set K and (ii) are situated at each knot of a three-dimensional Euclidean crystalline cubic structure.

Definition 1.2. *Deformed tensor products*

We consider a vector space V = {E(D, K), ⊗} where K is a commutative leaf, D indicates its dimension and ⊗ denotes the (classical and non deformed) tensor product. That product acts on pairs of elements arbitrarily chosen in V and, as usual (see any scholar book) :

$$\forall (\mathbf{q}_1, \mathbf{q}_2) \in V \times V : \mathbf{q}_1 \otimes \mathbf{q}_2 = q_1^{\alpha} \cdot q_2^{\beta} \cdot \mathbf{e}_{\alpha} \otimes \mathbf{e}_{\beta}$$

Nobody knows *a priori* if the result (the image) of a given tensor product is in V (the source) again :

$$\mathbf{q}_1 \otimes \mathbf{q}_2 \in V?$$

A pragmatic manner to be certain that the tensor product acts as an inner product is to equip V with a cube, A, and then to impose a relation of closure acting on the elements of its canonical basis Ω : (..., $\mathbf{e}_{\alpha}$, ...) :

$$\forall \mathbf{e}_{\alpha} \in \Omega : \exists A = \{A^{\chi}_{\alpha\beta} \in K \,|\, \mathbf{e}_{\alpha} \otimes \mathbf{e}_{\beta} = A^{\chi}_{\alpha\beta} \cdot \mathbf{e}_{\chi} \in V\},$$

Effectively, in imposing that relation :

$$\forall\,(\mathbf{q}_1,\mathbf{q}_2)\,\in\,V\,\times\,V\,:\,\mathbf{q}_1\,\otimes\,\mathbf{q}_2\,=\,q_1^\alpha\,.\,q_2^\beta\,.\,A^\chi_{\alpha\beta}\,.\,\mathbf{e}_\chi\,\in\,V,$$

Per convention, we say that the tensor product has been deformed by the (action of a) cube A and we relabel that tensor product as : $\otimes_A$.

$$\forall\,(\mathbf{q}_1,\mathbf{q}_2)\,\in\,V\,\times\,V\,:\,\otimes_A(\mathbf{q}_1,\mathbf{q}_2)\,=\,q_1^\alpha\,.\,q_2^\beta\,.\,A^\chi_{\alpha\beta}\,.\,\mathbf{e}_\chi\,\in\,V$$

Because V is isomorphic to its dual V*, it is also very usual and perhaps easier to work in V*. This can be done in introducing the well-known Dirac' convention ("bracket"). The deformed tensor product can also be understood as :

$$\forall\,(\mathbf{q}_1,\mathbf{q}_2)\,\in\,V\,\times\,V\,:\,|\otimes_A(\mathbf{q}_1,\mathbf{q}_2)>=\,|q_1^\alpha\,.\,q_2^\beta\,.\,A^\chi_{\alpha\beta}>\in\,M(D,1)$$

where M(D, 1) represents the set of matrices with one column and D rows.

Definition 1.3. *Decomposition*

Inspired by the concept of division, an operation that every child is supposed to learn during the first years of his/her life if he/she get the chance to visit a primary school, that theory introduces a concept of division concerning the dual representation of deformed tensor products.

In the common language, realizing a division in staying inside the set of natural numbers, N = {0, 1, 2, 3, ...}, is yielding a subset of pairs (M, R) in N^2; the first argument, M, is the main part of a given result whilst the second argument, R, is the residual part of that same result. For example : if we were dividing 13 by 3, we would write 13 = (4 x 3) + 1 and decode the manoeuvrer in saying that 4 is the main part whilst 1 is the residual one; hence, 13 :3 ≡ (M, R) = (4, 1).

This basic idea is extrapolated and applied now to deformed tensor products. Per convention, the first argument intervening in such a product is called a *projectile* and the second is called a *target*. In dividing a given deformed tensor product by its target, we expect to find a main part and a residual part too. Here, recalling considerations loaned to the concept of torsion, we suspect that this division will now result in a kind of decomposition that may reasonably be written :

$$(\mathbf{projectile},\mathbf{target})\in\,V^2\,:$$

$$|\otimes_A(\mathbf{projectile},\mathbf{target})>$$

$$=$$

$$[Main\,part]\,.\,|\mathbf{target}>\,+|\mathbf{residual\,part}>\in\,M(D,1)\,\sim\,K^D$$

$$[Main\,part]\,=\,[P]\,\in\,M(D,K),\,\mathbf{residual\,part}\,=\,\mathbf{z}\,\in\,E(D,\,K)$$

or more concisely :

$$(\mathbf{q}_1,\mathbf{q}_2)\in\,V^2\,:\,|\otimes_A(\mathbf{q}_1,\mathbf{q}_2)>=\,[P]\,.\,|\mathbf{q}_2>\,+|\mathbf{z}>$$

Remark 1.1. ***Postulat***

Recalling that mathematics is a motor for physics and that experiments in physics permanently exhibit the fact that our measurements are rarely precise, we shall postulate that, in general, a decomposition is only realized approximately. That means that it is realized with an error $\delta\mathbf{E}$ such that :

$$(\mathbf{q}_1,\mathbf{q}_2)\in\,V^2\,:\,\exists\,|\delta\mathbf{E}>=\,|\otimes_A(\mathbf{q}_1,\mathbf{q}_2)>\,-\{[P]\,.\,|\mathbf{q}_2>\,+|\mathbf{z}>\}$$

Definition 1.4. *Trivial decomposition*

Per convention we say that a decomposition is trivial when the residual part vanishes.

Proposition 1.1. *Existence of a trivial decomposition*

There are mathematical circumstances such that each image in V* of a given deformed tensor product accepts at least one trivial decomposition which we conventionally write :

$$(\mathbf{q}_1, \mathbf{q}_2) \in V^2 : \exists [{}_A\Phi(\mathbf{q}_1), \mathbf{0}] \,:\, | \otimes_A (\mathbf{q}_1, \mathbf{q}_2) >= {}_A\Phi(\mathbf{q}_1) \,.\, |\mathbf{q}_2 > + |\mathbf{0} >$$

Proof : Let write the deformed tensor product in extenso

$$\forall (\mathbf{q}_1, \mathbf{q}_2) \in V \times V \,:\, \otimes_A(\mathbf{q}_1, \mathbf{q}_2) = q_1^\alpha \,.\, q_2^\beta \,.\, A^\chi_{\alpha\beta} \,.\, \mathbf{e}_\chi \in V$$

Its image in V* is :

$$| \otimes_A (\mathbf{q}_1, \mathbf{q}_2)^\chi >= |(q_1^\alpha \,.\, q_2^\beta) \,.\, A^\chi_{\alpha\beta} >$$

If K is equiped with an associative and commutative multiplication, then :

$$| \otimes_A (\mathbf{q}_1, \mathbf{q}_2)^\chi >= |q_1^\alpha \,.\, (q_2^\beta \,.\, A^\chi_{\alpha\beta}) >= |q_1^\alpha \,.\, (A^\chi_{\alpha\beta} \,.\, q_2^\beta) >= |(q_1^\alpha \,.\, A^\chi_{\alpha\beta}) \,.\, q_2^\beta >$$

that can be recondensed in :

$$| \otimes_A (\mathbf{q}_1, \mathbf{q}_2) >= [A^\chi_{\alpha\beta} \,.\, q_1^\alpha] \,.\, |\mathbf{q}_2 >$$

Hence, there always exists at least one trivial decomposition denoted (${}_A\Phi(\mathbf{q}_1)$, $\mathbf{0}$) if {K, +, .} is a commutative ring. □

Remark 1.2. ***Characterizing a trivial decomposition***

Observing the postulat which has been proposed in remark 1.1 , it is a fact that a decomposition is trivial when its error of realization coincides with minus one time its residual part.

$$(\mathbf{q}_1, \mathbf{q}_2) \in V^2, \, \delta\mathbf{E} = -\mathbf{z} :$$

$$\Downarrow$$

$$| \otimes_A (\mathbf{q}_1, \mathbf{q}_2) > - \{[P] \,.\, |\mathbf{q}_2 > + |\mathbf{z} >\} = |\delta\mathbf{E} >= -|\mathbf{z} >$$

$$\Downarrow$$

$$| \otimes_A (\mathbf{q}_1, \mathbf{q}_2) > - [P] \,.\, |\mathbf{q}_2 >= |\mathbf{0} >$$

□

2 The procedure

2.1 Cubes and bilinear forms

The first necessary prerequisite for the development of the so-called *extrinsic method* is the existence of at least one non-degenerated bi-linear form, b, acting on pairs of elements in a vector space. Here, it is represented by a matrix [B] in M(D, K) with $|B| \neq 0$ and we consider a vector space W = {E(D, K), $\otimes_A$, [B]}.

Remark 2.1. ***Bilinear forms and metrics on W.***

In that paragraph, I roughly initiate a discussion concerning a combination link between a given cube and the existence of induced metrics.

Since W is equipped with one cube, there are [[01]; § 13.7, p. 331; in the French language]

$$A_{D^3}^{D^2} = \frac{D^3!}{(D^3 - D^2)!};\ D \in N - \{0, 1\}$$

theoretical possibilities to *arrange*, without repetition, D^2 components which have been extracted from the at most D^3 different components contained in the cube A. That means that we can find at most so many bi-linear forms for W. For example : if D = 2, then $A_8{}^4$ = 8.7.6.5 = 1 680 arrangements; if D = 3, then : $A_{27}{}^9$ = 27.26.25.24.23.22.21.20.19 = 1 700 755 056 000 arrangements; if D = 4, then : $A_{64}{}^{16}$ = 64.63.62.61.60.59.58.57.56.55.54.53.52.51.50.49 = 1,022134645914425. 10^{28} arrangements.

The last number is enorm. The concept of cube is illustrated in physics via the notion of connection. Due to intern symmetries, the effective number of different components can be smaller than the theoretical maximum; for example, the Levi-Civita connection is represented by a symmetric cube (that means that : $\Gamma_{\alpha\beta}{}^{\chi} = \Gamma_{\beta\alpha}{}^{\chi}$). Furthermore, effective models (example : isotropic metrics depending on the time) involve a set of at most eleven different expressions for the Christoffel's symbols of the second kind (see : star model in [[02]; chapter 44, p. 258; in the German language]) whilst the symmetric and diagonal metric itself has only at most four components. That means that, if we would like to extract bilinear forms (I did not say *metrics*) within that combinatorical context, we would have 11 ! :4 ! = 11.10.9.8.7.6.5 = 1 663 200 arrangements. This is a quite smaller number than the theoretical maximum.

On one side, we know since a long time that the Levi-Civita connection is metric compatible and that the correspondance between that connection and the metric is unique. On the other side, as intuitively expected, the combinatoric is yielding quite more bilinear forms than acceptable metrics. This is due to -at least two-well-identifiable facts : (i) not any bilinear form is a non-degenerated one; (ii) all components of a cube are not necessary different.

2.2 The scalar related to the projectile

Let consider any non-degenerated bilinear form b represented by the element [B] in M(D, K) :

$$(\mathbf{a}, \mathbf{b}) \in W^2 \xrightarrow{b} b(\mathbf{a}, \mathbf{b}) = < \mathbf{a}| \,.\, \{[B] \,.\, |\mathbf{b} >\} = < \mathbf{a} \,,\, \mathbf{b} >_{[B]} \in K$$

With that form, we can build an element in K (a scalar) which we say to be related to the projectile in calculating :

$$(\mathbf{q}_1, \mathbf{q}_2) \in W^2 \ : < \mathbf{q}_1 \,,\, \delta\mathbf{E} >_{[B]} = < \mathbf{q}_1 \,,\, | \otimes_A (\mathbf{q}_1, \mathbf{q}_2) > - \{[P] \,.\, |\mathbf{q}_2 > + |\mathbf{z} >\} >_{[B]}$$

Let suppose that there exists a function depending on the (components of the) projectile and such that :

$$\forall\, \mathbf{q}_1 \in W, \exists\, P_1 \in F(W, K) : P_1(\mathbf{q}_1) - P_1(\mathbf{0}) = < \mathbf{q}_1 \,,\, \delta\mathbf{E} >_{[B]}$$

This fact implies the existence of a scalar which can be interpreted as a polynomial form of degree two depending on the components of the projectile :

$$P_1(\mathbf{q}_1) = < \mathbf{q}_1 \,,\, | \otimes_A (\mathbf{q}_1, \mathbf{q}_2) > - \{[P] \,.\, |\mathbf{q}_2 > + |\mathbf{z} >\} >_{[B]} + P_1(\mathbf{0})$$

Since K is supposed to be equipped with a commutative multiplication, there is always at least one trivial decomposition for any given deformed tensor product :

$$P_1(\mathbf{q}_1)$$

$$=$$

$$< \mathbf{q}_1| \,.\, \{[B] \,.\, \{_A\Phi(\mathbf{q}_1) \,.\, |\mathbf{q}_2 >\}\} - < \mathbf{q}_1| \,.\, \{[B] \,.\, \{[P] \,.\, |\mathbf{q}_2 > + |\mathbf{z} >\}\} + P_1(\mathbf{0})$$

Let examine in which way that scalar may also be understood as the beginning of a Taylor - Mac Laurin development for the polynomial P_1 around the origin :

$$P_1(\mathbf{q}_1) = \frac{1}{2} \,.\, < \mathbf{q}_1| \,.\, \{[Hess P_1(\mathbf{0})] \,.\, |\mathbf{q}_1 >\} + < \mathbf{Grad}_{(\mathbf{q}_1)} P_1(\mathbf{0})| \,.\, |\mathbf{q}_1 > + P_1(\mathbf{0})$$

The scalar associated with the projectile coincides with such development if three conditions are simultaneously realized :

— Terms with degree two :

$$< \mathbf{q}_1| \,.\, \{[B] \,.\, \{_{[A]}\Phi(\mathbf{q}_1) \,.\, |\mathbf{q}_2 >\}\} = \frac{1}{2} \,.\, < \mathbf{q}_1| \,.\, \{[Hess P_1(\mathbf{0})] \,.\, |\mathbf{q}_1 >\} \quad (1)$$

$$\Downarrow$$

$$q_1^\chi \,.\, (A^\chi_{\alpha\beta} \,.\, q_2^\beta) \,.\, q_1^\alpha = \frac{1}{2} \,.\, q_1^\chi \,.\, \frac{\partial^2 P_1(\mathbf{q}_1 = \mathbf{0})}{\partial q_1^\chi \partial q_1^\alpha} \,.\, q_1^\alpha$$

$$\Downarrow$$

$$\forall\, \mathbf{q}_1 : [A^\chi_{\alpha\beta} \,.\, q_2^\beta] = \frac{1}{2} \,.\, [Hess_{\mathbf{q}_1} P_1(\mathbf{q}_1 = \mathbf{0})]$$

The left hand term must be managed carefuly ; although it is ressembling a trivial decomposition with the target as argument, it is neither that trivial decomposition nor its transposed. The exact link with that trivial decomposition depends on the properties of the cube A at hand. To avoid confusion in further development, we shall write :

$$[A^\chi_{\alpha\beta} \,.\, q_2^\beta] = {}_A\Psi(\mathbf{q}_2) \neq {}_A\Phi(\mathbf{q}_2)$$

As a matter of evidence, both matrices coincide when the cube is symmetric ; i.e. : $A_{\alpha\beta}{}^\chi = A_{\beta\alpha}{}^\chi$.

— Terms with degree one :

$$\mathbf{Grad}_{(\mathbf{q}_1)} P_1(\mathbf{0}) = -[B] \,.\, \{[P] \,.\, |\mathbf{q}_2 > + |\mathbf{z} >\} \quad (2)$$

— Term with degree zero :

$$0(3) = P_1(\mathbf{0}) \quad (3)$$

2.3 The scalar related to the target

With the same form, b, we can also build another element in K which we say to be related to the target in calculating :

$$(\mathbf{q}_1, \mathbf{q}_2) \in W^2 : < \mathbf{q}_2 \,,\, \delta\mathbf{E} >_{[B]} = < \mathbf{q}_2 \,,\, | \otimes_A (\mathbf{q}_1, \mathbf{q}_2) > - \{[P] \,.\, |\mathbf{q}_2 > + |\mathbf{z} >\} >_{[B]}$$

Let suppose that there exists a function depending on the (components of the) target and such that :

$$\forall\, \mathbf{q}_2 \in W, \exists\, P_2 \in F(W, K) : P_2(\mathbf{q}_2) - P_2(\mathbf{0}) = < \mathbf{q}_2 \,,\, \delta\mathbf{E} >_{[B]}$$

This existence implies the existence of a scalar which can be interpreted as a polynomial form of degree two depending on the components of a given target :

$$P_2(\mathbf{q}_2) \;=< \mathbf{q}_2\,,\,|\otimes_A(\mathbf{q}_1,\mathbf{q}_2)> \;-\{[P]\,.\,|\mathbf{q}_2>+|\mathbf{z}>\}>_{[B]}\;+P_2(\mathbf{0})$$

Since there is always at least one trivial decomposition for any given deformed tensor product, it follows :

$$P_2(\mathbf{q}_2) \;=< \mathbf{q}_2|\,.\,\{[B]\,.\,\{_A\Phi(\mathbf{q}_1)\;-\;[P]\}\,.\,|\mathbf{q}_2>\}-\;<\mathbf{q}_2|\,.\,\{[B]\,.\,|\mathbf{z}>\}\;+\;P_2(\mathbf{0})$$

Let examine in which way that scalar may also be understood as the beginning of a Taylor - Mac Laurin development for the polynomial P_2 around the origin :

$$P_2(\mathbf{q}_2)\;=\;\frac{1}{2}\,.\,<\mathbf{q}_2|\,.\,\{[HessP_2(\mathbf{0})]\,.\,|\mathbf{q}_2>\}+\,<\mathbf{Grad}_{(\mathbf{q}_2)}P_2(\mathbf{0})|\,.\,|\mathbf{q}_2>\,+P_2(\mathbf{0})$$

That eventuality occurs if three conditions are simultaneously realized :

— Terms of degree two :

$$[B]\,.\,\{_{[A]}\Phi(\mathbf{q}_1)\;-\;[P]\}\;=\;\frac{1}{2}\,.\,[HessP_2(\mathbf{0})] \tag{4}$$

— Terms of degree one :

$$\mathbf{Grad}_{(\mathbf{q}_2)}P_2(\mathbf{0})\;=\;-[B]\,.\,|\mathbf{z}> \tag{5}$$

— Term of degree zero :

$$0(3)\;=\;P_2(\mathbf{0})$$

2.4 Results.

If (i) the matrix [B] representing the bi-linear form at hand is not degenerated (i.e : if $|B| \neq 0$), and if (ii) the confrontation between the scalar related to a target and a Taylor - Mac Laurin development is meaningful, then the deformed tensor product involved in that procedure can be approximately decomposed. Indeed if :

$$|B|\;\neq\;0\;\Rightarrow\;\exists\,[B]^{-1}$$

and if the three necessary conditions are realized :

— Terms of degree two :

$$[P]\;=\;{}_A\Phi(\mathbf{q}_1)\;-\;\frac{1}{2}\,.\,[B]^{-1}\,.\,[HessP_2(\mathbf{0})] \tag{6}$$

— Terms of degree one :

$$|\mathbf{z}>=\;-[B]^{-1}\,.\,|\mathbf{Grad}_{(\mathbf{q}_2)}P_2(\mathbf{0})> \tag{7}$$

— Term of degree zero :

$$0(3)\;=\;P_2(\mathbf{0})$$

there is a set of plausible non-trivial decompositions ([P], $\mathbf{z}$) such that :

$$(\mathbf{q}_1,\mathbf{q}_2)\in W^2:\;|\delta\mathbf{E}>=\;|\otimes_A(\mathbf{q}_1,\mathbf{q}_2)>\;-\{[P]\,.\,|\mathbf{q}_2>+|\mathbf{z}>\} \tag{8}$$

2.5 Comments

The polynomials P_1 and P_2, their gradients and their classical Hessian matrices are the main ingredients of these acceptable decompositions. Let add some complementary comments. For the sake of generality, these polynomials have been chosen as if they were different ; but there are certainly some situations in physics allowing to repeat the method with a unique function :

$$P_1(\mathbf{q}_1) = L(\mathbf{q}_1, \mathbf{q}_2 = \mathbf{invariant})$$

$$P_2(\mathbf{q}_2) = L(\mathbf{q}_1 = \mathbf{invariant}, \mathbf{q}_2)$$

Since the same decomposition ([P], **z**) should be reached with both scalars, a confrontation between the conditions which have been obtained is theoretically meaningfull. Considering the equations (2), (5) and (4), we understand that that method can only be applied when :

$$\mathbf{Grad}_{(\mathbf{q}_2)} P_2(\mathbf{0}) - \mathbf{Grad}_{(\mathbf{q}_1)} P_1(\mathbf{0}) = \{[B] \,.\, {}_A\Phi(\mathbf{q}_1) - \frac{1}{2} \,.\, [HessP_2(\mathbf{0})]\} \,.\, |\mathbf{q}_2>$$

$$L(\mathbf{0}, \mathbf{0}) = 0(3)$$

This can be rewriten with more details in a generic coördinates system as :

$$\forall \alpha : \frac{\partial P_2(\mathbf{q}_2)}{\partial q_2^\alpha} - \frac{\partial P_1(\mathbf{q}_1)}{\partial q_1^\alpha} = b_{\alpha\chi} \,.\, A^\chi_{\delta\beta} \,.\, q_1^\delta \,.\, q_2^\beta - \frac{1}{2} \,.\, \frac{\partial^2 P_2(\mathbf{q}_2 = \mathbf{0})}{\partial q_2^\alpha \partial q_2^\beta} \,.\, q_2^\beta$$

In the context of that extrinsic method, the polynomials P_1 and P_2 are obviously involved in differential equations, respectively of the first and of the second order.

The method is said to be "extrinsic" because, in all cases, it needs the intervention of a non-degenerated bi-linear form which is an actor not intrinsically contained in the problematic dubbed as :

Definition 2.1. *The so-called (E) question*

$$(\mathbf{q}_1, \mathbf{q}_2) \in W^2 : \exists ? \,([P], \mathbf{z}) : |\otimes_A (\mathbf{q}_1, \mathbf{q}_2)> = \{[P] \,.\, |\mathbf{q}_2> + |\mathbf{z}>\}$$

2.6 Logical test

All mathematical methods have a domain of validity ; the extrinsic method too. The existence of an exact non-trivial decomposition annihilates the scalar related either to the projectile or to the target :

$$\exists\,([P], \mathbf{z}) : |\otimes_A (\mathbf{q}_1, \mathbf{q}_2)> - \{[P] \,.\, |\mathbf{q}_2> + |\mathbf{z}>\} = |\mathbf{0}>$$

$$\downarrow$$

$$\delta\mathbf{E} = \mathbf{0}$$

$$\downarrow$$

$$i = 1, 2 : < \mathbf{q}_i \,,\, \delta\mathbf{E} >_{[B]} = 0$$

But conversely, the vanishing of the scalar, for example related to the target, is corresponding to three plausible and different logical configurations :

$$< \mathbf{q}_2 \,,\, \delta\mathbf{E} >_{[B]} = 0$$

$$\downarrow$$

— **Configuration 1** - The target is null - the deformed tensor product is null - this situation is obviously meaningless here ;

$$\mathbf{q}_2 = \mathbf{0}$$

— **Configuration 2** - The non-trivial decomposition is exact :

$$\delta\mathbf{E} = \mathbf{0}$$

— **Configuration 3** - The target and the error are orthogonal but none of them vanishes :

$$\mathbf{q}_2 \neq \mathbf{0},\ \delta\mathbf{E} \neq \mathbf{0},\ < \mathbf{q}_2\,,\ \delta\mathbf{E} >_{[B]} = 0$$

3 Discussion.

3.1 Where, in physics, can the method be useful ?

Previous versions of that document have exposed the extrinsic method as an independent mathematical object. A better interconnection with others branches in physics would be appreciated.

A few years ago, I suggested to apply the extrinsic method to the covariant version of the Lorentz law in conditions preserving the Planck's constant ; see below the subsection 3.8-1. This was a one shot try without echo in the community. Per chance, I recently got a first encouraging sign in reading [03]. This is due to the fact that that article introduces a fluctuation operator, the formalism of which coincides with the difference between (i) the main part of the non-trivial decomposition which can be obtained with the extrinsic method for the deformed tensor product appearing in subsection 3.8-1 and (ii) its trivial part ; proof : compare [[03] ; p. 2, (5)] with equation (6). This remark opens a first door.

3.2 Completing the basics - recall.

For the completeness of this document I recall here the basics exposed in subsection 3.8-2. Up to now, the discussion is reduced to K = C, the set of complex numbers.

Definition 3.1. ***Deformed exterior product*** *In that theory and whatever the cube A is, inspired by the classical notion of exterior product, a deformed exterior product is :*

$$\forall\, \mathbf{a},\, \mathbf{b} \in E(D,\, C) :$$

$$\wedge_A(\mathbf{a}, \mathbf{b}) = \otimes_A(\mathbf{a},\, \mathbf{b}) - \otimes_A(\mathbf{b},\, \mathbf{a}) = \sum_{i,\, j,\, k\, =\, 0}^{D\, -\, 1} A^k_{ij} .\ (a^i .\ b^j - b^i .\ a^j) \,.\, \mathbf{e}_k$$

Obviously : (i) terms such that i = j vanish because C is equipped with a commutative multiplication and (ii) any deformed exterior product is an anti-symmetric operation. The definition can consequently be reformulated :

$$\forall\, \mathbf{a},\, \mathbf{b} \in E(D,\, C) :$$

$$\wedge_A(\mathbf{a}, \mathbf{b}) = \otimes_A(\mathbf{a},\, \mathbf{b}) - \otimes_A(\mathbf{b},\, \mathbf{a}) = \sum_{i<j}^{D\, -\, 1} (A^k_{ij} - A^k_{ji}) \,.\, (a^i \,.\, b^j - b^i \,.\, a^j) \,.\, \mathbf{e}_k$$

Definition 3.2. ***Symmetric cube*** *: A cube is symmetric relatively to its subscripts if :*

$$\forall\, i,\, j,\, k :\ A^k_{ij} \ -\ A^k_{ji} = 0$$

Remark 3.1. *A deformed exterior product which is built with a symmetric cube systematically vanishes.*

Definition 3.3. ***Antisymmetric cube :*** *A cube is antisymmetric relatively to its subscripts if :*

$$\forall\, i, j, k :\ A^k_{ij} \ +\ A^k_{ji} = 0$$

Definition 3.4. ***Deformed Lie product*** *: In that context and per convention, a deformed Lie product is only the half of a deformed exterior product which is built with the help of an anti-symmetric cube. When the discussion is developed in a space with dimension $D = 3$, that deformed Lie product is called a deformed cross product. The semantic will be justified a little bit later in the document.*

$$\forall\, \mathbf{a},\, \mathbf{b} \,\in\, E(D,C) :\ [\mathbf{a},\mathbf{b}]_{[A]} \ =\ \frac{1}{2} \,.\, \wedge_{[A]}\,(\mathbf{a},\mathbf{b}) \ =\ \sum_{i<j}^{D-1} \ A^k_{ij} \,.\, (a^i \,.\, b^j \ -\ b^i \,.\, a^j) \,.\, \mathbf{e}_k$$

Proposition 3.1. *Any deformed tensor product which is built upon an anti-symmetric cube is de facto a deformed Lie product which has been built with that cube.*

Démonstration. Let consider an anti-symmetric cube A :

$$\forall\,(\mathbf{a},\mathbf{b}) \,\in\, E^2(D,\, K),\, \forall\, A \,:\, A^k_{ij} \ +\ A^k_{ji} \ =\ 0$$

$$\otimes_A(\mathbf{a},\mathbf{b})$$

$$=$$

$$\sum_k (\sum_i \sum_j \ A^k_{ij} \,.\, a^i \,.\, b^j) \,.\, \mathbf{e}_k$$

$$=$$

$$\sum_k (\sum_{i<j} A^k_{ij} \,.\, a^i \,.\, b^j \ +\ \sum_{i=j} A^k_{ij} \,.\, a^i \,.\, b^j \ +\ \sum_{i>j} A^k_{ij} \,.\, a^i \,.\, b^j) \,.\, \mathbf{e}_k$$

$$=$$

$$\sum_k (\sum_{i<j} A^k_{ij} \,.\, a^i \,.\, b^j \ +\ 0 \,.\, a^i \,.\, b^j \ -\ \sum_{i<j} A^k_{ij} \,.\, a^j \,.\, b^i) \,.\, \mathbf{e}_k$$

$$=$$

$$\sum_k (\sum_{i<j} A^k_{ij} \,.\, (a^i \,.\, b^j \ -\ a^j \,.\, b^i) \,.\, \mathbf{e}_k$$

But, recall that any deformed exterior product is such that -where the cube A is any one :

$$\forall\, A,\, \forall\,(\mathbf{q}_1, \mathbf{q}_2) \,\in\, E^2(D,\, C) :$$

$$\wedge_A(\mathbf{q}_1,\mathbf{q}_2) \ =\ \otimes_A(\mathbf{q}_1,\mathbf{q}_2) \ -\ \otimes_A(\mathbf{q}_2,\mathbf{q}_1) \ =\ A^k_{ij} \,.\, (q^i_1 \,.\, q^j_2 \ -\ q^i_2 \,.\, q^j_1) \,.\, \mathbf{e}_k$$

All components can be organized in three subsets :

$$\{\sum_{i<j} A^k_{ij} \,.\, (q^i_1 \,.\, q^j_2 - q^i_2 \,.\, q^j_1)\} + \{\sum_{i=j} A^k_{ij} \,.\, (q^i_1 \,.\, q^j_2 - q^i_2 \,.\, q^j_1)\} + \{\sum_{i>j} A^k_{ij} \,.\, (q^i_1 \,.\, q^j_2 - q^i_2 \,.\, q^j_1)\}$$

Since the multiplication is a commutative operation for the elements in C : (i) the second subset vanishes and (ii) the terms appearing in the first and the third subsets are such that :

$$A^k_{12} \cdot (q^1_1 \cdot q^2_2 \; - \; q^1_2 \cdot q^2_1) \, ; \, A^k_{21} \cdot (q^2_1 \cdot q^1_2 \; - \; q^2_2 \cdot q^1_1)$$
$$A^k_{1j} \cdot (q^1_1 \cdot q^j_2 \; - \; q^1_2 \cdot q^j_1) \, ; \, A^k_{j1} \cdot (q^j_1 \cdot q^1_2 \; - \; q^j_2 \cdot q^1_1)$$
$$A^k_{1D} \cdot (q^1_1 \cdot q^D_2 \; - \; q^1_2 \cdot q^D_1) \, ; \, A^k_{D1} \cdot (q^D_1 \cdot q^1_2 \; - \; q^D_2 \cdot q^1_1)$$
$$etc.$$

In fine :

$$\forall \, A, \, \forall \, (\mathbf{q}_1, \mathbf{q}_2) \, \in \, E^2(D, \, K) \, : \, \wedge_A(\mathbf{q}_1, \mathbf{q}_2) \; = \; \sum_k \sum_{i<j} (A^k_{ij} - A^k_{ji}) \cdot (q^i_1 \cdot q^j_2 - q^i_2 \cdot q^j_1) \cdot \mathbf{e}_k$$

The result of this calculation is always an element in E(D, C). Each of the D components is a sum of 1 + 2 + ... + (N - 1) = N.(N - 1)/2 terms; instead of N^2 terms. Until now, the cube A is any one. If, furthermore, it is an anti-symmetric cube, then :

$$\forall \, A : \, A^k_{ij} \; + \; A^k_{ji} \; = \; 0, \, \forall \, (\mathbf{q}_1, \mathbf{q}_2) \, \in \, E^2(D, \, C) :$$
$$\wedge_A(\mathbf{q}_1, \mathbf{q}_2) \; = \; 2. \sum_{i<j} A^k_{ij} \cdot (q^i_1 \cdot q^j_2 \; - \; q^i_2 \cdot q^j_1) \cdot \mathbf{e}_k$$

From which, it is possible to get when $\mathbf{q}_1 = \mathbf{a}$ and $\mathbf{q}_2 = \mathbf{b}$:

$$\forall \, A : \, A^k_{ij} \; + \; A^k_{ji} \; = 0, \, \forall \, (\mathbf{a}, \mathbf{b}) \, \in \, E^2(D, \, C), \, \otimes_A(\mathbf{a}, \mathbf{b}) \; = \; \frac{1}{2} \wedge_A (\mathbf{a}, \mathbf{b}) \; = \; [\mathbf{a}, \mathbf{b}]_A$$

□

3.3 Confronting the methods.

Let do a resume of the situation. Let consider a given deformed Lie product in some D-dimensional space. The intrinsic method is able to furnish the main part of a non-trivial decomposition if D = 3; but, that method brings nothing else, neither about the residual part when D = 3 nor about the cases D greater than three. In opposition, any deformed tensor product (i) can be non-trivialy decomposed with the extrinsic method, whatever the dimension D of the space is, and (ii) is a Lie deformed product if the cube deforming it is antisymmetric (see previous demonstration). From these technical facts, I deduce that a confrontation between both methods in any three-dimensional space (D = 3) is possible when the discussion concerns the decomposition of deformed cross products. The label "deformed" cross product can be easily justified. Firstly :

Proposition 3.2. *In the context of this three-dimensional confrontation, in extenso : in a three dimensional space, any anti-symmetric cube can be reduced to an element in M(3, C).*

Démonstration. : Any cube can be considered as the mental superposition of three elements in M(3, C). When the cube is anti-symmetric, each diagonal vanishes and from the six off-diagonal terms only at most three can be different from the others. This is true for each of the three elements in M(3, C) constituting the cube at hand. Consequently, the initial cube is reduced to at most 3 × 3 = 9 different terms; they can be ordered inside an element in M(3, C).

$$\{\forall A \,|\, \forall \; i, \, j, \, k : \; A^k_{ij} + A^k_{ji} = 0\} \; \Rightarrow \; \{A \, \rightarrow \, [A] \, = \, \begin{bmatrix} A^1_{12} & A^2_{12} & A^3_{12} \\ A^1_{23} & A^2_{23} & A^3_{23} \\ A^1_{13} & A^2_{13} & A^3_{13} \end{bmatrix} \in M(3, \, C)\}$$

□

Definition 3.5. ***Deforming or topological matrix*** *: Per convention, a deforming matrix is the matrix which has been obtained after the anti-symetrization of some cube A.*

Secondly, it can be proved that :

$$|[\mathbf{a}, \mathbf{b}]_{[A]} >= [A]^t \, . \, [J] \, . \, |\mathbf{a} \wedge \mathbf{b} > \tag{9}$$

At a purely formal level, and as I shall explain it in details below, there are two kinds of confrontations :

1. The first kind confronts two polynomials of degree two depending on the components of the projectile (the first argument) : (i) the one resulting from the existence of the non-trivial decomposition itself ; (ii) the other one which is only the scalar associated with the projectile.
2. The second kind directly confronts the mains parts : (i) the one which is furnished by the intrinsic method and (ii) the other one which is given by the extrinsic method. Concretely, let consider the peculiar case [**q**, d**x**][A] as illustration ; the intrinsic main part :

$$[P]_{|A|} = |A| \, . \, [A]^t \, . \, [J] \, . \, \{\frac{1}{2} \, . \, [Hess_{(\mathbf{q},0)} \Lambda(\mathbf{q})] + \frac{1}{|A|} \, . \, {}_{[J]}\Phi({}_\Lambda\mathbf{s})\}, \, |A| = \pm 1$$

must be confronted with the extrinsic main part :

$$[P] = {}_A\Phi(\mathbf{q}) - \frac{1}{2} \, . \, [B]^{-1} \, . \, [HessP_2(d\mathbf{x} = \mathbf{0})]$$

There are two manners to do it :

(a) First interpretation :

i. Continuous deformation of the trivial part :

$${}_A\Phi(\mathbf{q}) = [A]^t \, . \, [J] \, . \, {}_J\Phi({}_\Lambda\mathbf{s}) \tag{10}$$

ii. Interdependence between the topological deformation [A] and the fundamental local bi-linear form [B] :

$$|A| \, . \, [A]^t \, . \, [J] = [B]^{-1} \tag{11}$$

iii. Interdependence between both Hessians :

$$[Hess_{(\mathbf{q},0)} \Lambda(\mathbf{q})] = [HessP_2(d\mathbf{x} = \mathbf{0})] \tag{12}$$

(b) Second interpretation :

i. Continuous deformation of the trivial part :

$${}_A\Phi(\mathbf{q}) = [A]^t \, . \, [J] \, . \, {}_J\Phi({}_\Lambda\mathbf{s})$$

ii. The Hessian of the proper intrinsic polynomial Λ is promoting the fundamental local bi-linear form [B] as Cartan' metric ; see explainations in [[04]] :

$$[Hess_{(\mathbf{q},0)} \Lambda(\mathbf{q})] = [B]^{-1} \tag{13}$$

iii. The topological deformation [A] is related to the extrinsic Hessian (P_2) when d$\mathbf{x} = \mathbf{0}$:

$$[HessP_2(d\mathbf{x} = \mathbf{0})] = |A| \, . \, [A]^t \, . \, [J] \tag{14}$$

3.4 Continuous deformation of the trivial part.

In both interpretations, there is a relation linking the trivial decomposition with an initial situation corresponding to the deforming matrix [J] and a projectile **q** that would coincide with the singular vector of the intrinsic polynomial Λ. The question is : "What can we learn from this relation ?" Let recall that :

$$[A] = \begin{bmatrix} A^1_{12} & A^2_{12} & A^3_{12} \\ A^1_{23} & A^2_{23} & A^3_{23} \\ A^1_{13} & A^2_{13} & A^3_{13} \end{bmatrix} \in M(3, C) \tag{15}$$

and :

$$[J]^t \,.\, [A] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} . \begin{bmatrix} A^1_{12} & A^2_{12} & A^3_{12} \\ A^1_{23} & A^2_{23} & A^3_{23} \\ A^1_{13} & A^2_{13} & A^3_{13} \end{bmatrix} = \begin{bmatrix} A^1_{23} & A^2_{23} & A^3_{23} \\ -A^1_{13} & -A^2_{13} & -A^3_{13} \\ A^1_{12} & A^2_{12} & A^3_{12} \end{bmatrix} \tag{16}$$

The transposed is :

$$[A]^t \,.\, [J] = \begin{bmatrix} A^1_{23} & -A^1_{13} & A^1_{12} \\ A^2_{23} & -A^2_{13} & A^2_{12} \\ A^3_{23} & -A^3_{13} & A^3_{12} \end{bmatrix} \tag{17}$$

These precisions allow to rewrite the equation (–) :

$$_A\Phi(\mathbf{q}) \tag{18}$$

$$=$$

$$\begin{bmatrix} A^1_{p1} \cdot q^p & A^1_{p2} \cdot q^p & A^1_{p3} \cdot q^p \\ A^2_{p1} \cdot q^p & A^2_{p2} \cdot q^p & A^2_{p3} \cdot q^p \\ A^3_{p1} \cdot q^p & A^3_{p2} \cdot q^p & A^3_{p3} \cdot q^p \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} A^1_{21} \cdot q^2 + A^1_{31} \cdot q^3 & A^1_{12} \cdot q^1 + A^1_{32} \cdot q^3 & A^1_{13} \cdot q^1 + A^1_{23} \cdot q^2 \\ A^2_{21} \cdot q^2 + A^2_{31} \cdot q^3 & A^2_{12} \cdot q^1 + A^2_{32} \cdot q^3 & A^2_{13} \cdot q^1 + A^2_{23} \cdot q^2 \\ A^3_{21} \cdot q^2 + A^3_{31} \cdot q^3 & A^3_{12} \cdot q^1 + A^3_{32} \cdot q^3 & A^3_{13} \cdot q^1 + A^3_{23} \cdot q^2 \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} A^1_{23} & -A^1_{13} & A^1_{12} \\ A^2_{23} & -A^2_{13} & A^2_{12} \\ A^3_{23} & -A^3_{13} & A^3_{12} \end{bmatrix} . \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}$$

There are nine relations ; for the first row :

$$A^1_{21} \,.\, q^2 + A^1_{31} \,.\, q^3 = -A^1_{13} \,.\, s_3 - A^1_{12} \,.\, s_2 \tag{19}$$

$$A^2_{21} \,.\, q^2 + A^2_{31} \,.\, q^3 = -A^2_{13} \,.\, s_3 - A^2_{12} \,.\, s_2$$

$$A^3_{21} \,.\, q^2 + A^3_{31} \,.\, q^3 = -A^3_{13} \,.\, s_3 - A^3_{12} \,.\, s_2$$

For the second row :

$$A^1_{12} \,.\, q^1 + A^1_{32} \,.\, q^3 = -A^1_{23} \,.\, s_3 + A^1_{12} \,.\, s_1$$

$$A^2_{12} \,.\, q^1 + A^2_{32} \,.\, q^3 = -A^2_{23} \,.\, s_3 + A^2_{12} \,.\, s_1$$

$$A^3_{12} \,.\, q^1 + A^3_{32} \,.\, q^3 = -A^3_{23} \,.\, s_3 + A^3_{12} \,.\, s_1$$

For the third row :

$$A^1_{13} \,.\, q^1 + A^1_{23} \,.\, q^2 = A^1_{23} \,.\, s_2 + A^1_{13} \,.\, s_1$$

$$A^1_{13} \,.\, q^1 \,+\, A^1_{23} \,.\, q^2 \,=\, A^2_{23} \,.\, s_2 \,+\, A^2_{13} \,.\, s_1$$
$$A^1_{13} \,.\, q^1 \,+\, A^1_{23} \,.\, q^2 \,=\, A^3_{23} \,.\, s_2 \,+\, A^3_{13} \,.\, s_1$$

Any deforming (equiv. topological) matrix [A] can be interpreted as (i) either a line of three vectors, when the components of each of them are disposed in a row; (ii) or a row of three vectors, when the components of each of them are disposed in a line. Indeed, each (3-3) matrix contains implicitly six vectors because :

$$[A] \,=\, \begin{bmatrix} A^1_{12} & A^2_{12} & A^3_{12} \\ A^1_{23} & A^2_{23} & A^3_{23} \\ A^1_{13} & A^2_{13} & A^3_{13} \end{bmatrix}$$

can always be visualy decoded as :

$$[A] \,=\, [|\mathbf{a}_1 >,\, |\mathbf{a}_2 >,\, |\mathbf{a}_3 >] \tag{20}$$

with :

$$|\mathbf{a}_\eta > \equiv \left| \begin{matrix} A^\eta_{12} \\ A^\eta_{23} \\ A^\eta_{13} \end{matrix} \right\rangle ;\ \eta \,=\, 1,\, 2,\, 3.$$

or as :

$$[A] \,=\, \begin{bmatrix} |\mathbf{a}^3 > \\ |\mathbf{a}^1 > \\ |\mathbf{a}^2 > \end{bmatrix} \tag{21}$$

All components (i) are elements in K and (ii) belong to at least two vectors. This fact justifies the semantic discribing the situation : "There is a six-pack of intricated vectors which is associated to any (3-3) matrix in M(3, K)". I also introduce the :

Definition 3.6. ***Sum of the components for a given vector.***

The application "Sum of the components for a given vector" has its source in E(3, K) and its image in K; it works as explained here :

$$\forall\, \mathbf{v} \,\in\, E(3\, K) \xrightarrow{\oplus} \oplus(\mathbf{v}) \,=\, \mathbf{v}^\oplus \,=\, \sum_{p\,=\,1}^{3} v^p$$

The nine prior relations can then be summed as :

$$-\mathbf{a}^{3\oplus} \,.\, q^2 \,-\, \mathbf{a}^{2\oplus} \,.\, q^3 \,=\, -\mathbf{a}^{3\oplus} \,.\, s_2 \,-\, \mathbf{a}^{2\oplus} \,.\, s_3$$
$$\mathbf{a}^{3\oplus} \,.\, q^1 \,-\, \mathbf{a}^{1\oplus} \,.\, q^3 \,=\, \mathbf{a}^{3\oplus} \,.\, s_1 \,-\, \mathbf{a}^{1\oplus} \,.\, s_3$$
$$\mathbf{a}^{2\oplus} \,.\, q^1 \,+\, \mathbf{a}^{1\oplus} \,.\, q^2 \,=\, \mathbf{a}^{1\oplus} \,.\, s_2 \,+\, \mathbf{a}^{2\oplus} \,.\, s_1$$

Definition 3.7. ***The topological vector.***

The three sums $\mathbf{a}^{1\oplus}$, - $\mathbf{a}^{2\oplus}$, $\mathbf{a}^{3\oplus}$ can be understood as the components of a vector $\mathbf{a}$([A]).

Example 3.1. ***The Euclidean topological vector***

It is extremely easy to verify that the classical (syn. : non-deformed) cross product is equivalent to a cross product which would have been deformed by the matrix [J]. As a direct consequence :

$$[A] \,=\, [J] \,\Rightarrow\, {}_{[J]}\mathbf{a} \,:\, (1, 1, 1) \tag{22}$$

This definition justifies the rewritting of the nine relations as :

$$_J\Phi(_{[A]}\mathbf{a}) \,.\, |\mathbf{q}> = \;_J\Phi(_{[A]}\mathbf{a}) \,.\, |_\Lambda\mathbf{s}>$$

This is implying :

$$_{[A]}\mathbf{a} \wedge \mathbf{q} \;=\; _{[A]}\mathbf{a} \wedge \,_\Lambda\mathbf{s}$$

And the :

Theorem 3.1. ***Admissible projectile.***

Any deformed cross product [**q**, d**x**][A] can be decomposed non-trivially with a fine-tuned usage of two mathematical methods, the intrinsic and the extrinsic, when the polynomial which is associated with that decomposition is a proper one. The fine-tuning is imposing a precise formalism for the projectile :

$$\mathbf{q} \;=\; _\Lambda\mathbf{s} \,+\, \mu \,.\, _{[A]}\mathbf{a},\; \forall\, \mu \,\in\, K \tag{23}$$

Here, $_\Lambda\mathbf{s}$ is the singular vector of the proper polynomial Λ and **a**[A] is the local topological vector.

Corollary 3.1. ***The Euclidean enigma.***

Let realize a trivial projection of any admissible projectile and state that :

$$_A\Phi(\mathbf{q}) \;=\; _A\Phi(_\Lambda\mathbf{s}) \,+\, \mu \,.\, _A\Phi(_{[A]}\mathbf{a}) \tag{24}$$

Let consider the equation (10) again :

$$_A\Phi(\mathbf{q}) \;=\; [A]^t \,.\, [J] \,.\, _J\Phi(_\Lambda\mathbf{s})$$

Let remark that if the deformed cross product is a classical one, then [A] = [J] and :

$$_A\Phi(\mathbf{q}) \;=\; _J\Phi(\mathbf{q}) \;=\; [J]^t \,.\, [J] \,.\, _J\Phi(_\Lambda\mathbf{s}) \;=\; _J\Phi(_\Lambda\mathbf{s}) \,\Rightarrow\, \{\mathbf{q} \,=\, _\Lambda\mathbf{s}\} \tag{25}$$

The unique admissible projectile in a classical context is the singular vector of the proper polynomial Λ. But, let inject this result into the trivial projection of any admissible projectile in focusing on the classical context and state that it imposes :

$$\{[A] \,=\, [J]\} \,\Rightarrow\, \{(24) \,\rightarrow\, \mu \,.\, _J\Phi(_{[J]}\mathbf{a}) \,=\, [0]\} \tag{26}$$

Since the Euclidean topological vector is not null (see example 3.1), within a classical context, there is a unique admissible projectile in coincidence with the singular vector of the proper intrinsic polynomial Λ and the paramater μ must vanish.

$$\{(26) \,\cap\, _{[J]}\mathbf{a} \,\neq\, \mathbf{0}\} \,\Rightarrow\, \{\mu \,=\, 0\}$$

This result is in frontal opposition with the intuitive existence of the trivial decomposition $_J\Phi(\mathbf{q})$ for any cross product [**q**, **...**][J] ; see proposition 1.1.

Corollary 3.2. ***Domain of definition of the theory - Initial conditions.***

It indirectly suggests that the domain of definition of the confrontation between both methods is limited to cross products in which the first argument (the projectile) is the singular vector for some proper polynomial Λ ; i.e. : [$_\Lambda$**s**, **...**][J]. Since any cross product is an anti-symmetric operation, the constraint concerns the second argument as well ; and the initial situations described by this theory probably are :

$$[_\Lambda\mathbf{s},\, _\Psi\mathbf{s}][J] \,\rightarrow\, [\mathbf{q},\, d\mathbf{x}][A]$$

$$\mathbf{q} = {}_{\Lambda}\mathbf{s} + \mu \, . \, {}_{[A]}\mathbf{a} \, ; \, d\mathbf{x} = {}_{\Psi}\mathbf{s} + \rho \, . \, {}_{[A]}\mathbf{a}$$

Provided the three-dimensional Euclidean space is supposed to correspond to the initial geometrical state, the theory of the (E) question is able to study the deformations and the decompositions of a limited family of classical cross products in which each argument is the singular vector for a proper polynomial of degree two :

$$\{[G] = Id_3, \, {}_{\Lambda}\mathbf{s} \wedge {}_{\Psi}\mathbf{s}\}$$

$$\downarrow$$

$$\{\forall \, [G], \, [\mathbf{q} = {}_{\Lambda}\mathbf{s} + \mu \, . \, {}_{[A]}\mathbf{a}, \, \delta\mathbf{x} = {}_{\Psi}\mathbf{s} + \rho \, . \, {}_{[A]}\mathbf{a}][A]\}$$

Since, in general :

$$|[\mathbf{q}, \, \delta\mathbf{x}][A] >= [A]^t \, . \, [J] \, . \, |\mathbf{q} \wedge \delta\mathbf{x} >$$

The theory considers classical cross products of which the deformations in any geometry is :

$$|[\mathbf{q}, \, \delta\mathbf{x}][A] >= [A]^t \, . \, [J] \, . \, \{|{}_{\Lambda}\mathbf{s} \wedge {}_{\Psi}\mathbf{s} > + (\rho \, . \, {}_{\Lambda}\mathbf{s} - \mu \, . \, {}_{\Psi}\mathbf{s}) \wedge {}_{[A]}\mathbf{a}\} \tag{27}$$

Example 3.2. ***Explosion***

In our every day geometry, I consider a singular vector related to a given polynomial Λ. The theory allows to start a discussion with the null vector :

$$\{[G] = Id_3, \, {}_{\Lambda}\mathbf{s} \wedge {}_{\Lambda}\mathbf{s} = \mathbf{0}\}$$

But, it also allows deformations of the null vector yielding the new vector :

$$|[\mathbf{q}, \, \delta\mathbf{x}][A] >= [A]^t \, . \, [J] \, . \, \{(\rho - \mu) \, . \, {}_{\Lambda}\mathbf{s} \wedge {}_{[A]}\mathbf{a}\}$$

Here, the projectile and the target belong to the same polynomial Λ but, each of them "feels" the deformation in a particular way ; hence the existence of two parameters μ and ρ.

The deformed vector (cross product) results from a peculiar type of interaction involving (i) the singular vector on one side and (ii) the topological (deforming) matrix [A] and its induced topological vector **a**[A] on the other side. This is a totally funny result and a specificity of that strange theory.

3.5 The ADM formalism

The measurement of lengths plays a crucial role in physics. Since Riemann' work, it is a usual to start a discussion with :

$$(\delta s)^2 = g_{\lambda\mu} \, . \, \delta x^{\lambda} \, . \, \delta x^{\mu}$$

It is a polynomial of degree two depending on the four components of d**x**. There are four manners to write it as a polynomial of degree two depending on three of these four components. For simplicity, I shall start with the 3 + 1 split :

$$(\delta s)^2 = g_{ab} \, . \, \delta x^a \, . \, \delta x^b + \{(g_{0a} + g_{a0}) \, . \, \delta x^0\} \, . \, \delta x^a + g_{00} \, . \, (\delta x^0)^2$$

It can be confronted with the canonical ADM formalism [[05] ; p. 7, (3.9), (3.10) and (3.11)] :

$$(\delta s)^2 = g_{ab} \, . \, \delta x^a \, . \, \delta x^b + \{2 \, . \, N_a \, . \, \delta x^0\} \, . \, \delta x^a - (N^2 - N_a \, . \, N^a) \, . \, (\delta x^0)^2$$

Any way, that split exhibits a polynomial of degree two, say Ψ to avoid confusions with the inverse cross product, depending on the three spatial components of d$\mathbf{x}$:

$$d_{ab} \, = \, g_{ab}$$

$$d_a \, = \, 2 \, . \, N_a \, . \, \delta x^0$$

$$d \, = \, - \, (N^2 \, - \, N_a \, . \, N^a) \, . \, (\delta x^0)^2$$

This fact suggests the existence of a whole family of deformed cross products $[\delta\mathbf{x}, \boldsymbol{...}]_{[A]}$. Leaving the problematic of the three others possible splits for later, let recall the main results of the intrinsic method ; in peculiar :

$$Hess_{(\delta\mathbf{x}, \, 0)} P(\delta\mathbf{x}) \, = \, {}^{(3)}[G] \, + \, {}^{(3)}[G]^t$$

$$|_{\mathbf{\Psi}}\mathbf{s} >= \, -2 \, . \, \delta x^0 \, . \, \{{}^{(3)}[G] \, + \, {}^{(3)}[G]^t\}^{-1} \, . \, |\mathbf{N}^* >$$

For coherence with previous paragraphs, let reduce the discussion once more time to non-degenerated spatial symmetric metrics. There are a set of deformed cross products admitting non-trivial decompositions :

$$|\delta\mathbf{x} \wedge \mathbf{q} >= \, [Q]_{|A|} \, . \, |\mathbf{q} > \, + |\mathbf{residual\,part} >; \, |A| \, = \, \pm 1$$

with :

$$[Q]_{|A|} \, = \, |A| \, . \, \{[A]^t \, . \, [J]\} \, . \, \{{}^{(3)}[G] \, + \, \frac{\delta x^0}{|A|} \, . \, {}_{[J]}\Phi([G]^{-1} \, . \, |\mathbf{N}^* >)\}$$

Remark 3.2. ***The coefficients of degree one of the polynomial.***

In general :

$$d_1$$

$$=$$

$$A^1_{12} \, . \, (q_{31} \, . \, q_{23} \, - \, q_{21} \, . \, q_{33}) \, + \, A^2_{12} \, . \, (q_{33} \, . \, q_{11} \, - \, q_{31} \, . \, q_{13}) \, + \, A^3_{12} \, . \, (q_{21} \, . \, q_{13} \, - \, q_{11} \, . \, q_{23})$$

$$+$$

$$A^1_{13} \, . \, (q_{21} \, . \, q_{32} \, - \, q_{22} \, . \, q_{31}) \, + \, A^2_{13} \, . \, (q_{31} \, . \, q_{12} \, - \, q_{11} \, . \, q_{32}) \, + \, A^3_{13} \, . \, (q_{11} \, . \, q_{22} \, - \, q_{21} \, . \, q_{12})$$

$$d_2$$

$$=$$

$$A^1_{12} \, . \, (q_{32} \, . \, q_{23} \, - \, q_{22} \, . \, q_{33}) \, + \, A^2_{12} \, . \, (q_{33} \, . \, q_{12} \, - \, q_{32} \, . \, q_{13}) \, + \, A^3_{12} \, . \, (q_{22} \, . \, q_{13} \, - \, q_{12} \, . \, q_{23})$$

$$+$$

$$A^1_{23} \, . \, (q_{21} \, . \, q_{32} \, - \, q_{22} \, . \, q_{31}) \, + \, A^2_{23} \, . \, (q_{31} \, . \, q_{12} \, - \, q_{11} \, . \, q_{32}) \, + \, A^3_{23} \, . \, (q_{11} \, . \, q_{22} \, - \, q_{21} \, . \, q_{12})$$

$$d_3$$

$$=$$

$$A^1_{13} \, . \, (q_{32} \, . \, q_{23} \, - \, q_{22} \, . \, q_{33}) \, + \, A^2_{13} \, . \, (q_{33} \, . \, q_{12} \, - \, q_{32} \, . \, q_{13}) \, + \, A^3_{13} \, . \, (q_{22} \, . \, q_{13} \, - \, q_{12} \, . \, q_{23})$$

$$+$$

$$A^1_{23} \, . \, (q_{21} \, . \, q_{33} \, - \, q_{23} \, . \, q_{31}) \, + \, A^2_{23} \, . \, (q_{31} \, . \, q_{13} \, - \, q_{11} \, . \, q_{33}) \, + \, A^3_{23} \, . \, (q_{11} \, . \, q_{23} \, - \, q_{21} \, . \, q_{13})$$

Remark 3.3. ***The coefficients of degree one of the polynomial for a classical cross product.***

In peculiar, if [A] = [J] :

$$d_1$$
$$=$$
$$0 \,.\, (q_{31} \cdot q_{23} \,-\, q_{21} \cdot q_{33}) \,+\, 0 \,.\, (q_{33} \cdot q_{11} \,-\, q_{31} \cdot q_{13}) \,+\, 1 \,.\, (q_{21} \cdot q_{13} \,-\, q_{11} \cdot q_{23})$$
$$+$$
$$0 \,.\, (q_{21} \cdot q_{32} \,-\, q_{22} \cdot q_{31}) \,-\, 1 \,.\, (q_{31} \cdot q_{12} \,-\, q_{11} \cdot q_{32}) \,+\, 0 \,.\, (q_{11} \cdot q_{22} \,-\, q_{21} \cdot q_{12})$$
$$=$$
$$(q_{21} \cdot q_{13} \,-\, q_{11} \cdot q_{23}) \,-\, (q_{31} \cdot q_{12} \,-\, q_{11} \cdot q_{32})$$
$$=$$
$$q_{11} \,.\, (q_{32} \,-\, q_{23}) \,+\, (q_{21} \cdot q_{13} \,-\, q_{31} \cdot q_{12})$$

$$d_2$$
$$=$$
$$0 \,.\, (q_{32} \cdot q_{23} \,-\, q_{22} \cdot q_{33}) \,+\, 0 \,.\, (q_{33} \cdot q_{12} \,-\, q_{32} \cdot q_{13}) \,+\, 1 \,.\, (q_{22} \cdot q_{13} \,-\, q_{12} \cdot q_{23})$$
$$+$$
$$1 \,.\, (q_{21} \cdot q_{32} \,-\, q_{22} \cdot q_{31}) \,+\, 0 \,.\, (q_{31} \cdot q_{12} \,-\, q_{11} \cdot q_{32}) \,+\, 0 \,.\, (q_{11} \cdot q_{22} \,-\, q_{21} \cdot q_{12})$$
$$=$$
$$(q_{22} \cdot q_{13} \,-\, q_{12} \cdot q_{23}) \,+\, (q_{21} \cdot q_{32} \,-\, q_{22} \cdot q_{31})$$
$$=$$
$$q_{22} \,.\, (q_{13} \,-\, q_{31}) \,+\, (q_{21} \cdot q_{32} \,-\, q_{12} \cdot q_{23})$$

$$d_3$$
$$=$$
$$0 \,.\, (q_{32} \cdot q_{23} \,-\, q_{22} \cdot q_{33}) \,-\, 1 \,.\, (q_{33} \cdot q_{12} \,-\, q_{32} \cdot q_{13}) \,+\, 0 \,.\, (q_{22} \cdot q_{13} \,-\, q_{12} \cdot q_{23})$$
$$+$$
$$1 \,.\, (q_{21} \cdot q_{33} \,-\, q_{23} \cdot q_{31}) \,+\, 0 \,.\, (q_{31} \cdot q_{13} \,-\, q_{11} \cdot q_{33}) \,+\, 0 \,.\, (q_{11} \cdot q_{23} \,-\, q_{21} \cdot q_{13})$$
$$=$$
$$-\,1 \,.\, (q_{33} \cdot q_{12} \,-\, q_{32} \cdot q_{13}) \,+\, (q_{21} \cdot q_{33} \,-\, q_{23} \cdot q_{31})$$
$$=$$
$$q_{33} \,.\, (q_{21} \,-\, q_{12}) \,+\, (q_{32} \cdot q_{13} \,-\, q_{23} \cdot q_{31})$$

Remark 3.4. ***The main part of the decomposition for a classical cross product.***

In peculiar, if [A] = [J], then |A| = -1 and the main part writes :

$$[Q]_{-1} \,=\, -^{(3)}[G] \,-\, {}_{[J]}\Phi({}_{\Psi}\mathbf{s})$$

with :

$$|{}_{\Psi}\mathbf{s}> = \,-\delta x^0 \,.\, {}^{(3)}[G]^{-1} \,.\, |\mathbf{N}^* >$$

That main part is in general not exactly symmetric. Nevertheless, when the lapse of time becomes increasingly small ($\delta x^0 \to 0$), the anti-symmetric part of [Q] vanishes. By the way, let remark that the lapse vector $\mathbf{N}^*$ vanishes anyway when the main part is symmetric. Hence, for any infinitesimal lapse of time, the intrinsic non-trivial decomposition reaches the limit :

$$\{[A] = [J],\, \delta x^0 \,\to\, 0\} \,\Rightarrow\, \{[Q]_{-1} \,=\, -^{(3)}[G]\}$$

At the Euclidean limit, this relation is :

$$\{[A] = [J],\, \delta x^0 \,\to\, 0,\, [G] \,=\, Id_3\} \,\Rightarrow\, \{[Q]_{-1} \,=\, -Id_3\}$$

It amongst to the same to write :

$$\{[A] = [J],\, \delta x^0 \,\to\, 0,\, [G] \,=\, Id_3\} \,\Rightarrow\, \{\delta\mathbf{x} \wedge \mathbf{q} \,=\, -\mathbf{q} \,+\, \mathbf{residual\,part}\}$$

This relation is another characteristic of the Euclidean enigma within the theory of the (E) question. I mean : the intuitively expected trivial decomposition ${}_{J}\Phi(\delta\mathbf{x})$ does not occur when one adopts the inner logic of that theory. Otherwise, the previous equation has nothing revolutionnary since any three-dimensional vector (the residual part) can always be decomposed into two parts : one along the direction of the target ($\mathbf{q}$) and one along a direction orthogonal to the former ($\delta\mathbf{x} \wedge \mathbf{q}$).

3.6 Cartan's spinors and the Euclidean enigma.

In that three-dimensional Euclidean context, let state that :

$$< \mathbf{q},\, \delta\mathbf{x} \,\wedge\, \mathbf{q} > = \,-(\mathbf{q})^2 + \,< \mathbf{q},\, \mathbf{residual\,part} > = \,0$$

This relation can be validated in only two cases :

— Either the residual part coincides with the target ; but, in that case, the target must also be proportional to the projectile at hand, precisely $\delta\mathbf{x}$, and the discussion concerns a null vector.

— Or the target $\mathbf{q}$ is an isotropic vector (see definition and concept in [[06]]) which must be orthogonal to the residual part.

Hence, the Euclidean enigma introduces de facto spinors into the discussion and forces to focuse on K = C, the set of all complex numbers. The concept of spinor in a three-dimensional Euclidean space is introduced in Cartan' work via the relations [[06] ; §42, pp. 41-42] :

$$s^1 \,=\, (\eta^0)^2 \,+\, (\eta^1)^2$$

$$s^2 \,=\, i \,.\, \{(\eta^0)^2 \,-\, (\eta^1)^2\}$$

$$s^3 \,=\, 2i \,.\, \eta^0 \,.\, \eta^1$$

for which it is easy to state that :

$$||\mathbf{s}||^2 \,=\, (s^1)^2 \,+\, (s^2)^2 \,+\, (s^3)^2$$

$$\downarrow$$

$$||\mathbf{s}||^2 = \{(\eta^0)^4 + (\eta^1)^4 + 2\,.\,(\eta^0)^2\,.\,(\eta^1)^2\} - \{(\eta^0)^4 + (\eta^1)^4 - 2\,.\,(\eta^0)^2\,.\,(\eta^1)^2\} - 4\,.\,(\eta^0)^2\,.\,(\eta^1)^2$$

$$\downarrow$$

$$||\mathbf{s}||^2 = 0$$

These relations prove that it is possible to build an application with a source (s^1, s^2, s^3) in the part of C^3 containing all isotropic vectors and an image (η^0, η^1) in the part of C^2 containing elements which are dubbed with the name "spinor" ; and conversely. A priori, Cartan' historical application can be generalized in writing :

$$s^1(\eta^0\,,\,\eta^1) = a^1_{00}\,.\,(\eta^0)^2 + a^1_{01}\,.\,\eta^0\,.\,\eta^1 + a^1_{11}\,.\,(\eta^1)^2$$

$$s^2(\eta^0\,,\,\eta^1) = a^2_{00}\,.\,(\eta^0)^2 + a^2_{01}\,.\,\eta^0\,.\,\eta^1 + a^2_{11}\,.\,(\eta^1)^2$$

$$s^3(\eta^0\,,\,\eta^1) = a^3_{00}\,.\,(\eta^0)^2 + a^3_{01}\,.\,\eta^0\,.\,\eta^1 + a^3_{11}\,.\,(\eta^1)^2$$

and in giving the conditions insuring the vanishing of the Euclidean norm of $\mathbf{s}$:

1. $$(\eta^0)^4 \;:\; \{(a^1_{00})^2 + (a^2_{00})^2 + (a^3_{00})^2\} = 0$$
2. $$(\eta^0)^3\,.\,(\eta^1) \;:\; a^1_{00}\,.\,a^1_{01} + a^2_{00}\,.\,a^2_{01} + a^3_{00}\,.\,a^3_{01} = 0$$
3. $$(\eta^0)^2\,.\,(\eta^1)^2 \;:\; a^1_{00}\,.\,a^1_{11} + a^2_{00}\,.\,a^2_{11} + a^3_{00}\,.\,a^3_{11} + \{(a^1_{01})^2 + (a^2_{01})^2 + (a^3_{01})^2\} = 0$$
4. $$(\eta^0)\,.\,(\eta^1)^3 \;:\; a^1_{01}\,.\,a^1_{11} + a^2_{01}\,.\,a^2_{11} + a^3_{01}\,.\,a^3_{11} = 0$$
5. $$(\eta^1)^4\,terms \;:\; \{(a^1_{11})^2 + (a^2_{11})^2 + (a^3_{11})^2\} = 0$$

That discussion obviously introduces the matrix :

$$[W] = \begin{bmatrix} a^1_{00} & a^1_{01} & a^1_{11} \\ a^2_{00} & a^2_{01} & a^2_{11} \\ a^3_{00} & a^3_{01} & a^3_{11} \end{bmatrix}$$

Here again, that matrix can be understood as :

$$[W] = [|\mathbf{w}_1>, |\mathbf{w}_2>, |\mathbf{w}_3>]$$

And this visualisation allows a rephrasing of the five conditions :

1. $$(\eta^0)^4 \;:\; ||\mathbf{w}_1||^2 = 0$$
2. $$(\eta^0)^3\,.\,(\eta^1) \;:< \mathbf{w}_1\,,\,\mathbf{w}_2 >_{Id_3} = 0$$
3. $$(\eta^0)^2\,.\,(\eta^1)^2 \;:< \mathbf{w}_1\,,\,\mathbf{w}_3 >_{Id_3} + ||\mathbf{w}_2||^2 = 0$$
4. $$(\eta^0)\,.\,(\eta^1)^3 \;:< \mathbf{w}_2\,,\,\mathbf{w}_3 >_{Id_3} = 0$$
5. $$(\eta^1)^4 \;:\; ||\mathbf{w}_3||^2 = 0$$

With that generalization, a spinor can be seen as the generator of three intersecting ellipsoids (one for each component of $\mathbf{s}$). Three vectors in E(3, C), precisely $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$, contain the caracteristics of these figures; two of them are isotropic vectors : $\mathbf{w}_1$ and $\mathbf{w}_3$. The remaining one, $\mathbf{w}_2$, is orthogonal to the two isotropic vectors and its Euclidean norm is minus one time the scalar product between the two isotropic vectors.

Since any cross product is anti-symmetric operation, in that Euclidean context, it must be possible to write :

$$\{[A] = [J],\, \delta x^0 \to 0,\, [G] = Id_3\} \Rightarrow \{\mathbf{q} \wedge \delta\mathbf{x} = -\delta\mathbf{x} + \textbf{another residual part}\}$$

But, once again, this relation is meaningful if and only if the new target, i.e. : $\delta\mathbf{x}$, is also an isotropic vector orthogonal to that other residual part. The discussion is coherent if and only if the sum of four vectors vanishes :

$$-\delta\mathbf{x} - \mathbf{q} + \textbf{residual part} + \textbf{another residual part} = \mathbf{0}$$

That discussion also indicates that the matrices :

$$[W]^- = [|\mathbf{q}>, |\textbf{residual part}>, |\delta\mathbf{x}>]$$

and :

$$[W]^+ = [|\delta\mathbf{x}>, |\textbf{another residual part}>, |\mathbf{q}>]$$

where $\mathbf{q}$ and $\delta\mathbf{x}$ are isotropic vectors in E(3, C), play an important role in the understanding of the Euclidean enigma. For now, assembling all previous results concerning the cross product in an Euclidean geometry, the theory imposes :

$$\{\mu = 0,\, \mathbf{q} = {}_\Lambda\mathbf{s}\} \cap \{(\mathbf{q})^2 = 0\} \Rightarrow \{\mu = 0,\, (\mathbf{q})^2 = ({}_\Lambda\mathbf{s})^2 = 0\}$$

The projectile is the singular vector of some proper polynomial of degree two (I call it Λ) and the projectile, hence that singular vector as well, must be isotropic. It can be treaten with the help of the generalization of Cartan' proposition :

$$|{}_\Lambda\mathbf{s}> = [W]\,.\,|\eta>$$

$$|\eta> = \left|\begin{array}{c} (\eta^0)^2 \\ \eta^0\,.\,\eta^1 \\ (\eta^1)^2 \end{array}\right\rangle$$

The coincidence between the projectile and the singular vector and the isotropy impose a clear constraint :

$$({}_\Lambda\mathbf{s})^2 = ({}_\Lambda\mathbf{s},\, {}_\Lambda\mathbf{s}) = <_\Lambda \mathbf{s},\, {}_\Lambda\mathbf{s}>_{Id_3} = <\eta\,.\,[W]^t\,.\,[W]\,.\,|\eta> = 0$$

It is easy to check that :

$$[W]^t\,.\,[W] = \begin{bmatrix} s^1 & s^2 & s^3 \\ z^1 & z^2 & z^3 \\ \delta x^1 & \delta x^2 & \delta x^3 \end{bmatrix} . \begin{bmatrix} s^1 & z^1 & \delta x^1 \\ s^2 & z^2 & \delta x^2 \\ s^3 & z^3 & \delta x^3 \end{bmatrix} = \begin{bmatrix} ({}_\Lambda\mathbf{s})^2 & (\mathbf{s},\, \mathbf{z}) & (\mathbf{s},\, \delta\mathbf{x}) \\ (\mathbf{z},\, \mathbf{x}) & (\mathbf{z})^2 & (\mathbf{z},\, \delta\mathbf{x}) \\ (\delta\mathbf{x},\, \mathbf{s}) & (\delta\mathbf{x},\, \mathbf{z}) & (\delta\mathbf{x})^2 \end{bmatrix} \quad (28)$$

Definition 3.8. ***Fundamental triad.***

In that theory, a fundamental triad $\mathbf{T}$ is a set of three vectors involved in the resolution of the (E) question at the Euclidean limit; for example here : $(\mathbf{q}, \mathbf{z}, \mathrm{d}\mathbf{x})$ and in general (**projectile**, **residual part**, **target**). Since that concept is related to the existence of a classical product, it is meaningful to introduce the inverse fundamental trial (**target**, **another residual part**, **projectile**) and to recall the relation of coherence connecting the four actors :

$$-\,\mathbf{projectile}\,-\,\mathbf{target}\,+\,\mathbf{residual\,part}\,+\,\mathbf{another\,residual\,part}\,=\,\mathbf{0} \quad (29)$$

This relation suggests the existence of an intuitive link between the four actors which are involved in the treatment of the (E) question at the Euclidean limit and a generic tetrahedron.

Remark 3.5. ***Fundamental triad and the representation of spinors in a three-dimensional Euclidean space.***

If a triad (**projectile**, **residual part**, **target**) -any one, not obligatorily a fundamental one- generates a matrix [W] connecting an isotropic vector **s** (which, at this stage, is not necessarily a singular vector of some proper polynomial) and the vector η, then the five conditions impose :

$$[W]^t\,.\,[W]\,=\,T_2(<\,...,\,...\,>_{Id_3})(\mathbf{T},\,\mathbf{T})\,=\,(\mathbf{z})^2\,.\,\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad (30)$$

And, in that case :

$$({}_{\Lambda}\mathbf{s})^2\,=\,(\eta\,.\,[W]^t,\,[W]\,.\,|\eta)\,=\,-(\eta^0\,.\,\eta^1\,.\,\mathbf{z})^2$$

The vanishing of the norm is obtained when :

1. Either one of the two components of the spinor vanihes whatever the residual part is ; in general :

$$\begin{aligned} \eta^1 &= 0 \\ s^1(\eta^0\,,\,\eta^1) &= a^1_{00}\,.\,(\eta^0)^2 \\ s^2(\eta^0\,,\,\eta^1) &= a^2_{00}\,.\,(\eta^0)^2 \\ s^3(\eta^0\,,\,\eta^1) &= a^3_{00}\,.\,(\eta^0)^2 \\ \eta^0 &= 0 \\ s^1(\eta^0\,,\,\eta^1) &= a^1_{11}\,.\,(\eta^1)^2 \\ s^2(\eta^0\,,\,\eta^1) &= a^2_{11}\,.\,(\eta^1)^2 \\ s^3(\eta^0\,,\,\eta^1) &= a^3_{11}\,.\,(\eta^1)^2 \end{aligned}$$

   In peculiar here, for the triad ($\mathbf{q}$, $\mathbf{z}$, $\delta\mathbf{x}$) involved in the decomposition $|\mathbf{q}\times\delta\mathbf{x}> = [\mathrm{P}].|\delta\mathbf{x}> + |\mathbf{z}>$:

$$\eta^0\,=\,0,\;\mathbf{s}\,=\,(\eta^1)^2\,.\,\delta\mathbf{x}$$

$$\eta^1\,=\,0,\;\mathbf{s}\,=\,(\eta^0)^2\,.\,\mathbf{q}$$

   If the triad is a fundamental one, then the coherence imposes $\eta^0 = \pm 1$.

2. Or the residual part **z** is itself an isotropic vector whatever the spinor η is :

$$(\mathbf{z})^2\,=\,0$$

   A fundamental triad of which the three vectors are isotropic vectors is called an isotropic fundamental triad.

Remark 3.6. ***Double isotropic fundamental triad***

A special configutation is realized when the four actors are forming two isotropic fundamental triads because, in that case :

$$(\mathbf{projectile})^2 + (\mathbf{target})^2 + (\mathbf{residual\,part})^2 + (\mathbf{another\,residual\,part})^2 = \mathbf{0} \tag{31}$$

Provided the components of each actor would be the inverse of the radius of a given circle, a visual confrontation with the first property, equation (–), may very roughly evocate Descartes Circles formula or, quite more interestingly, the formula concerning D + 2 spheres in D-dimensional spaces ; see [[07]] for a short presentation and more details.

3.7 Conclusion

In that document, I have introduced a mathematical method, the initial purpose of which is the discovery of non-trivial decompositions for deformed tensor products. Its essence is based on the belief that this kind of decomposition exists but, is never exactly realized due to an inherent principle of uncertainty governing the nature ; e.g. : Heisenberg' one. Hence, the method gives approximative results.

Fortunately, any deformed tensor product which has been built with the help of an anti-symmetric cube is a deformed Lie product and, in three-dimensional spaces, the imprecisions can be eliminated ; for example : in confronting the main part of a generic extrinsic decomposition with the one which has been obtained with the intrinsic method.

In doing so, although I considerably reduce the domain of definition of that theory :

1. I can write the allowed arguments which can be involved in the deformed cross products ;
2. I discover that they all depend on (i) a singular vector and (ii) on a topological vector that can be built with the help of the local deforming (topological) matrix.
3. I also discover that that theory contains what I have called an Euclidean enigma which is characterized by the unexpected facts :
   — The unique allowed arguments in that geometrical context are singular vectors of some proper polynomials ; that means : the parameter of proportionality acting as a modulation on the topological vector vanishes althoug the later does not.

   — Contre-intuitively, the trivial decomposition (classical rotation) is not realized and Cartan's spinors must be introduced into the discussion.

   — There are the specific relations :

$$\delta\mathbf{x} \wedge \mathbf{q} = -\mathbf{q} + \mathbf{another\,residual\,part} \,;\, \delta\mathbf{x} = {}_{\Psi}\mathbf{s}$$

$$\mathbf{q} \wedge \delta\mathbf{x} = -\delta\mathbf{x} + \mathbf{residual\,part} \,;\, \mathbf{q} = {}_{\Lambda}\mathbf{s}$$

   They can be summerized with :

$${}_{\Psi}\mathbf{s} \wedge {}_{\Lambda}\mathbf{s} = -{}_{\Lambda}\mathbf{s} + \mathbf{another\,residual\,part}$$

$${}_{\Lambda}\mathbf{s} \wedge {}_{\Psi}\mathbf{s} = -{}_{\Psi}\mathbf{s} + \mathbf{residual\,part}$$

The sum is yielding :

$$-{}_\Lambda\mathbf{s} + {}_\Psi\mathbf{s} + \mathbf{residual\,part} + \mathbf{another\,residual\,part} = \mathbf{0}$$

which is just a peculiar representation of equation (28). This condition can be realized in diverse manners but, there is one which is particularly simple :

$${}_\Lambda\mathbf{s} = \mathbf{residual\,part}\,;\, \mathbf{another\,residual\,part} = {}_\Psi\mathbf{s}$$

In that case, the calibration which has been proposed between both mathematical methods reduces the theory to the study of cross products involving two singular vectors which, in a three-dimensional Euclidean geometry, are equal to their difference ; precisely :

$${}_\Lambda\mathbf{s} \wedge {}_\Psi\mathbf{s} = {}_\Lambda\mathbf{s} - {}_\Psi\mathbf{s}$$

Furthermore each singular vector must be isotropic :

$$({}_\Lambda\mathbf{s})^2 = ({}_\Psi\mathbf{s})^2 = 0$$

These situations open a chapter that focuse the attention on the evolution of a singular vector related to a unique proper polynomial :

$${}_\Lambda\mathbf{s}(t_2) \wedge {}_\Lambda\mathbf{s}(t_1) = {}_\Lambda\mathbf{s}(t_2) - {}_\Lambda\mathbf{s}(t_1) = d_\Lambda\mathbf{s}$$

This exploration should be continued. For example, it would be interesting to get a precise description for the evolutions of the initial cross products ; the equation (27) is a starting point for that quest. An extrapolation in spaces with a dimension greater than three would also be a progress.

3.8 Personal contributions

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