MATRICIAL DERIVATIONS

INTRODUCING A TEENAGER INTUITIVE IDEA

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Contents

1 Preliminaries

1.1 Concepts

1.1.1 Basics

- C is the set of all complex numbers; it is an algebra.

- E(D, C) denotes a D dimensional (D $\geq$ 2) vector space built on C; that means that any element **u** in that space is refered to a canonical basis Ω: ($\mathbf{e}_1$, $\mathbf{e}_2$, ..., $\mathbf{e}_\alpha$, ..., $\mathbf{e}_D$) and can be written $\mathbf{u} = \sum_\alpha \mathrm{u}^\alpha \ . \ \mathbf{e}_\alpha$ with u^α in C forall α in Ind(D) = {1, 2, ..., D}.

- In that theory, a rang-D cube is a mathematical object that should be thought as a regular cubic structure of which each knot is occupied by a number.

- $\mathrm{C}_{(D-D-D)}$ is the set of all cubes of which the knots are occupied by complex numbers.

- The symbol $\otimes$ denotes the classical tensor product acting on two vectors.

- **Deformed tensor product**:
  Let A be an element in $\mathrm{C}_{(D-D-D)}$, $\otimes_A$ is a tensor product which has been deformed by the cube A. Concretely that deformed tensor product acts on pairs of vectors in the following manner:

$$\mathbf{u},\, \mathbf{w} \,\in\, E(D,\, C):$$

$$\otimes_A(\mathbf{u},\, \mathbf{w}) \,=\, \sum_\chi A^\chi_{\alpha\beta}\,.\,u^\alpha\,.\,w^\beta\,.\,\mathbf{e}_\chi \,\in\, E(D,\, C)$$

- **Pseudo-deformed exterior product**:

$$\forall\,(\mathbf{q}_1, \mathbf{q}_2) \in E^2(D,\,K)\,:$$

$$\wedge_A(\mathbf{q}_1, \mathbf{q}_2) = \otimes_A(\mathbf{q}_1, \mathbf{q}_2) - \otimes_A(\mathbf{q}_2, \mathbf{q}_1) = A^k_{ij} \,.\, (q^i_1 \,.\, q^j_2 - q^i_2 \,.\, q^j_1) \,.\, \mathbf{e}_k$$

The components of that new vector can allways be separated into three subsets:

$$\{\sum_{i<j} A^k_{ij} \,.\, (q^i_1 \,.\, q^j_2 - q^i_2 \,.\, q^j_1)\}$$
$$+\,\{\sum_{i=j} A^k_{ij} \,.\, (q^i_1 \,.\, q^j_2 - q^i_2 \,.\, q^j_1)\}$$
$$+\,\{\sum_{i>j} A^k_{ij} \,.\, (q^i_1 \,.\, q^j_2 - q^i_2 \,.\, q^j_1)\}$$

The second sum vanishes because the discussion is developped with elements in C. The remaining terms have the generic formalism:

$$A^k_{12} \,.\, (q^1_1 \,.\, q^2_2 - q^1_2 \,.\, q^2_1)\,;\; A^k_{21} \,.\, (q^2_1 \,.\, q^1_2 - q^2_2 \,.\, q^1_1)$$
$$A^k_{1j} \,.\, (q^1_1 \,.\, q^j_2 - q^1_2 \,.\, q^j_1)\,;\; A^k_{j1} \,.\, (q^j_1 \,.\, q^1_2 - q^j_2 \,.\, q^1_1)$$
$$A^k_{1D} \,.\, (q^1_1 \,.\, q^D_2 - q^1_2 \,.\, q^D_1)\,;\; A^k_{D1} \,.\, (q^D_1 \,.\, q^1_2 - q^D_2 \,.\, q^1_1)$$
$$etc.$$

So that:

$$\forall\,A,\,\forall\,(\mathbf{q}_1, \mathbf{q}_2) \in E^2(D,\,K)\,:$$
$$\wedge_A(\mathbf{q}_1, \mathbf{q}_2) = \sum_{i<j} (A^k_{ij} - A^k_{ji}) \,.\, (q^i_1 \,.\, q^j_2 - q^i_2 \,.\, q^j_1) \,.\, \mathbf{e}_k$$

The result is a new vector in E(D, K); each of its D components is the sum of 1 + 2 + ... + (D - 1) = D.(D - 1)/2 terms.

- **Deformed Lie product**:
In that theory, a deformed Lie product is a pseudo-deformed exterior product which is built with the help of an anti-symmetric cube:

$$\forall\,A\,:\, A^k_{ji} + A^k_{ij} = 0,\,\forall\,(\mathbf{q}_1, \mathbf{q}_2) \in E^2(D,\,K)\,:$$

$$[\mathbf{q}_1, \mathbf{q}_2]_A = \frac{1}{2} \,.\, \wedge_A(\mathbf{q}_1, \mathbf{q}_2) = \sum_{i<j} A^k_{ij} \,.\, (q^i_1 \,.\, q^j_2 - q^i_2 \,.\, q^j_1) \,.\, \mathbf{e}_k$$

- **Euclidean scalar product**: The symbol <..., ...> denotes a classical euclidean scalar product acting on pairs of elements taken in E(D, C):

$$(\mathbf{u},\, \mathbf{w}) \,\in\, E^2(D,\, C) : < \mathbf{u},\, \mathbf{w} >= \sum_\alpha u^\alpha \,.\, w^\alpha$$

- The symbol $V_D = \{E(D, C), \otimes_A\}$ represents a vector space E(D, C) equipped with the deformed tensor product $\otimes_A$.

- An **isotropic vector** in V_D is a non-vanishing vector **u** in E(D, C) such that:

$$\otimes_A(\mathbf{u},\, \mathbf{u}) \,=\, \mathbf{0}$$

- A **Jacobi's territory** is a set of elements (**u**, **v**, **w**) in E^3(D, C) such that:

$$\otimes_A(\mathbf{u},\, \otimes_A(\mathbf{v},\, \mathbf{w})) \,=\, \otimes_A(\otimes_A(\mathbf{u},\, \mathbf{v}),\, \mathbf{w}) + \otimes_A(\mathbf{v},\, \otimes_A(\mathbf{u},\, \mathbf{w}))$$

1.1.2 Dual interpretation

Any deformed tensor product is represented by a set of D components in C; hence, it has at least and de facto a representation which is an element in C^D. Furthermore, each of its D components can be understood as the result (in extenso: the concrete representation in C) of a function f^χ acting on E^2(D, C); χ being in Ind(D).

$$\forall\, \chi \,\in\, Ind(D),\, f^\chi :\, E^2(D,\, C) \,\rightarrow\, C$$

$$(\mathbf{u},\, \mathbf{w}) \,\rightarrow\, f^\chi(\mathbf{u},\, \mathbf{w}) \,=\, A^\chi_{\alpha\beta} \,.\, u^\alpha \,.\, w^\beta$$

As a matter of facts, each function f^χ is bilinear and characterized by a (D-D) matrix $[_\chi A]$ in M(D, C). The superposition of these D matrices is one of the representations of (the cube) A.

1.1.3 Pseudo-Casimir associated with a (D-D) matrix

Any element in M(D, C) can be visually decoded as either the superposition of D lines or the juxtaposition of D rows. Each line can be interpreted as the dual representation $<_\chi \mathbf{a}^\alpha|$ of some vector $_\chi \mathbf{a}^\alpha$ while each row can be interpreted as the dual representation $|_\chi \mathbf{a}_\beta>$ of some vector $_\chi \mathbf{a}_\beta$.

Example: D = 3. The matrix

$$[_\chi A] = \begin{bmatrix} A^\chi_{11} & A^\chi_{12} & A^\chi_{13} \\ A^\chi_{21} & A^\chi_{22} & A^\chi_{23} \\ A^\chi_{31} & A^\chi_{32} & A^\chi_{33} \end{bmatrix}$$

can be understood either as:

$$[_\chi A] = [|_\chi \mathbf{a}_1 >, |_\chi \mathbf{a}_2 >, |_\chi \mathbf{a}_3 >]$$

with:

$$|\mathbf{a}_\eta > \equiv \left| \begin{matrix} A^\chi_{1\eta} \\ A^\chi_{2\eta} \\ A^\chi_{1\eta} \end{matrix} \right\rangle ;\ \eta = 1,\, 2,\, 3.$$

or, in the same vein, as:

$$[_\chi A] = \begin{bmatrix} |_\chi \mathbf{a}^1 > \\ |_\chi \mathbf{a}^2 > \\ |_\chi \mathbf{a}^3 > \end{bmatrix}$$

This decodage allows the construction of a *pseudo-Casimir* for each matrix $[_\chi \mathrm{A}]$:

$$\forall \chi \in Ind(D),\ Cas : M(D,\, C) \to C$$

$$Cas([_\chi A]) = z_\chi = \sum_\alpha <_\chi \mathbf{a}^\alpha,\, _\chi \mathbf{a}_\alpha >$$

1.1.4 Pseudo-Casimir vector associated with a cube.

The existence of any deformed tensor product is related to the one of at least one cube A. That cube can be decomposed in a superposition of D matrices and a pseudo-Casimir can be associated with each of them. Hence, a set of D complex numbers is automatically associated with that cube; they may represent the components of some vector **z** in E(D, C). I call it the *pseudo-Casimir vector* associated with the cube A. As a matter of fact, that vector only depends on that cube; therefore, a deformed tensor product can be understood as a special type of interaction between that pseudo-Casimir vector and a pair of vectors in E(D, C).

$$\mathbf{Cas} : C_{(D-D-D)} \to E(D, C)$$
$$A \to \mathbf{Cas}(A) \equiv (..., z_\chi = \sum_\alpha <_\chi \mathbf{a}^\alpha, {}_\chi\mathbf{a}_\alpha >, ...)$$

1.2 Trivial decompositions

1.2.1 Existence

Proposition:
A deformed product always has at least one trivial decomposition in C^D.

Proof for the deformed tensor product:
Due to the dual interpretation:

$$\otimes_A(\mathbf{q}_1, \mathbf{q}_2) \in E(D, C) \to |\sum_{i,j} A^k_{ij} \,.\, q^i_1 \,.\, q^j_2 > \in C^D$$

The dual representation can always be put under a mixed formalism:

$$| \otimes_A (\mathbf{q}_1, \mathbf{q}_2) > = {}_A\Phi(\mathbf{q}_1) \,.\, |\mathbf{q}_2 >$$

That representation is a mixed one in that sense that the writing mixes a matrix, ${}_A\Phi(\mathbf{q}_1)$, and the dual representation $|\mathbf{q}_2>$ of the target. The matrix and the vector zero form a pair $({}_A\Phi(\mathbf{q}_1), \mathbf{0})$ which is the so-called most trivial decomposition of the deformed tensor product at hand.

Proof for the deformed exterior product:
Due to the dual interpretation:

$$\wedge_A(\mathbf{q}_1, \mathbf{q}_2) \in E(D, C) \to |\sum_{i<j} (A^k_{ij} - A^k_{ji}) \,.\, (q^i_1 \,.\, q^j_2 - q^i_2 \,.\, q^j_1) > \in C^D$$

Therefore:

$$| \wedge_A (\mathbf{q}_1, \mathbf{q}_2) >$$
$$=$$
$$\sum_{i<j} (A^k_{ij} \, - \, A^k_{ji}) \, . \, (q^i_1 \, . \, q^j_2 \, - \, q^i_2 \, . \, q^j_1)$$
$$=$$
$$| \sum_{i<j} A^k_{ij} \, . \, (q^i_1 \, . \, q^j_2 \, - \, q^i_2 \, . \, q^j_1) > \, - \, | \sum_{i<j} A^k_{ji} \, . \, (q^i_1 \, . \, q^j_2 \, - \, q^i_2 \, . \, q^j_1) >$$
$$=$$
$$| \sum_{i<j} A^k_{ij} \, . \, (q^i_1 \, . \, q^j_2 \, - \, q^i_2 \, . \, q^j_1) > \, + \, | \sum_{i>j} A^k_{ij} \, . \, (q^i_1 \, . \, q^j_2 \, - \, q^i_2 \, . \, q^j_1) >$$

At the end, this can indeed again be written as:

$$|[\mathbf{q}_1, \mathbf{q}_2]_A >= \, _A\Phi(\mathbf{q}_1) \, . \, |\mathbf{q}_2 >$$

1.2.2 Surjection

Let consider V_D and ask the question if a given element [M] in M(D, C) can sometimes be the representation of a trivial decomposition?

$$[M] \, \in \, M(D, \, K), \, \exists ? \, \mathbf{q}_1 \, \in \, V_D :$$
$$[M] \, = \, [m_{\lambda\mu}] \, = \, [A^\lambda_{\chi\mu} \, . \, q^\chi_1] \, = \, _A\Phi(\mathbf{q}_1)$$

A cube A and a matrix [M] being given, the following scalars can always be calculated:

$$\forall \, \epsilon \, \in \, Ind(D) : \, _1q_\epsilon \, = \, \frac{1}{2} \, . \, \sum_\lambda \sum_\mu A^\lambda_{\epsilon\mu} \, . \, m_{\lambda\mu}$$

Let suppose the existence of a non-degenerated element [G] in M(D, K) such that:

$$_1q^\delta \, = \, \sum_\epsilon g^{\delta\epsilon} \, . \, _1q_\epsilon \, = \, \frac{1}{2} \, . \, \sum_\epsilon g^{\delta\epsilon} \, . \, \sum_\lambda \sum_\mu A^\lambda_{\epsilon\mu} \, . \, m_{\lambda\mu}$$

It is then possible to calculate:

$$\sum_\delta A^\gamma_{\delta\beta} \, . \, _1q^\delta \, = \, \frac{1}{2} \, . \, \sum_\delta A^\gamma_{\delta\beta} \, . \, \sum_\epsilon g^{\delta\epsilon} \, . \, \sum_\lambda \sum_\mu A^\lambda_{\epsilon\mu} \, . \, m_{\lambda\mu}$$

These products are the components of [M] when:

$$m_{\gamma\beta} = \frac{1}{2} \cdot \sum_{\delta} A^{\gamma}_{\delta\beta} \cdot \sum_{\epsilon} g^{\delta\epsilon} \cdot \sum_{\lambda} \sum_{\mu} A^{\lambda}_{\epsilon\mu} \cdot m_{\lambda\mu}$$

Let remark that:

$$m_{\gamma\beta} = \sum_{\lambda} \sum_{\mu} \delta^{\lambda}_{\gamma} \cdot \delta^{\mu}_{\beta} \cdot m_{\lambda\mu}$$

The condition writes:

$$\delta^{\lambda}_{\gamma} \cdot \delta^{\mu}_{\beta} = \frac{1}{2} \cdot \sum_{\delta} \sum_{\epsilon} A^{\gamma}_{\delta\beta} \cdot g^{\delta\epsilon} \cdot A^{\lambda}_{\epsilon\mu}$$

Conversely, if a given matrix [M] in M(D, C) is a trivial decomposition, then there exists at least one element ${}_1\mathbf{q}$ in V_D such that:

$$[M] = {}_A\Phi(\mathbf{q}_1)$$

If, furthermore, that element belongs to the subset of V_D of which the covariant components have the formalism:

$${}_1q_{\epsilon} = \frac{1}{2} \cdot \sum_{\lambda} \sum_{\mu} A^{\lambda}_{\epsilon\mu} \cdot m_{\lambda\mu}$$

Then:

$${}_1q_{\epsilon} = \frac{1}{2} \cdot \sum_{\lambda} \sum_{\mu} A^{\lambda}_{\epsilon\mu} \cdot A^{\lambda}_{\delta\mu} \cdot {}_1q^{\delta}$$

And there exists an element [G] in M(D, C) allowing the conversion between the co- and the contravariant components of that vector:

$$g_{\epsilon\delta} = \frac{1}{2} \cdot \sum_{\lambda} \sum_{\mu} A^{\lambda}_{\epsilon\mu} \cdot A^{\lambda}_{\delta\mu}$$

1.2.3 Example: D = 4; Lorentz transformations.

There is a natural illustration of the previous surjection in mathematical physics for spaces with the dimension D = 4 through the exponent appearing in the representation of a generic Lorentz transformation:

$$[\Lambda] = exp^{-\frac{1}{2} \cdot \sum_{\lambda} \sum_{\mu} \omega_{\mu\nu} \cdot [J^{\mu\nu}]}$$

The generators of that generic transformation are the matrices $[J^{\lambda\mu}]$ corresponding to three translations (boosts) and three rotations. The ponderation of the generators is realized through the components of a skew-symmetric matrix $[\omega]$:

$$[\omega] = \begin{bmatrix} 0 & \beta^1 & \beta^2 & \beta^3 \\ -\beta^1 & 0 & -\theta^3 & \theta^2 \\ -\beta^2 & \theta^3 & 0 & -\theta^1 \\ -\beta^3 & -\theta^2 & \theta^1 & 0 \end{bmatrix},$$

A first approximation allows:

$$[\Lambda] = Id_4 - \frac{1}{2} . \omega_{\mu\nu} . [J^{\mu\nu}] + ...$$

In any frame where that approximative relation can be diagonalized:

$$\forall \alpha = 0, 1, 2, 3 : \Lambda_\alpha - 1 = \frac{1}{2} . \sum_\mu \sum_\nu \omega_{\mu\nu} . J^{\mu\nu}_\alpha$$

There is a clear formal analogy with the relation describing a specific subset of V_4 allowing the definition of a surjection ${}_A\Phi$ when:

$$A \equiv J$$

$$[M] \equiv [\omega]$$

Three arguments allow to think that both equivalences are systematically realized:

1. The doubt surrounding the diagonalization can always be eliminated because the job of Lorentz transformations is to preserve the metrics. This is written:

   $$[G] = [\Lambda]^t . [G] . [\Lambda]$$

   And that type of relation makes it possible to diagonalize the matrices (give more precisions please).

2. Six diagonalized generators[1] contain twenty-four scalars (6 × 4). Any cube A in $C_{(4-4-4)}$ of which the components have anti-symmetric subscripts (${A^\chi}_{\alpha\beta} + {A^\chi}_{\beta\alpha} = 0$) contains also only twenty-four distinct scalars.

[1]Each generator is a (4-4) matrix in M(4, C).

3. The correspondance between a set of six weights and the six components of a skew-symmetric matrix is just a matter of convention and always realizable.

Hence, any Lorentz transformation can be related to a vector:

$$\mathbf{q} : (..., \Lambda_\alpha - 1, ...) = (..., \frac{1}{2} . \sum_\mu \sum_\nu \omega_{\mu\nu} . J^{\mu\nu}_\alpha, ...); \alpha = 0, 1, 2, 3.$$

And the *weighting matrix* $[\omega]$ can be interpreted as a trivial decomposition of $\otimes_J(\mathbf{q}, ...)$:

$$[\omega] = {}_J\Phi(\mathbf{q})$$

In that case, a preserved and non-degenerated metric [G] is related to the generators of Lorentz transformations via the relation:

$$\delta^\lambda_\gamma . \delta^\mu_\beta = \frac{1}{2} . \sum_\delta \sum_\epsilon J^{\delta\beta}_\gamma . g^{\delta\epsilon} . J^{\epsilon\mu}_\lambda$$

1.2.4 Multiplicative morphism

Let consider the relation:

$${}_A\Phi(\otimes_A(\mathbf{u}, \mathbf{v})) = {}_A\Phi(\mathbf{u}) . {}_A\Phi(\mathbf{v})$$

It is equivalent to:

$$A^\lambda_{\chi\mu} . (A^\chi_{\alpha\beta} . u^\alpha . v^\beta) = (A^\lambda_{\alpha\gamma} . u^\alpha) . (A^\gamma_{\beta\mu} . v^\beta)$$

or:

$$(A^\lambda_{\chi\mu} . A^\chi_{\alpha\beta} - A^\lambda_{\alpha\gamma} . A^\gamma_{\beta\mu}) . u^\alpha . v^\beta = 0$$

There are two types of situations which are compatible with the existence of that multiplicative morphism:

- **type 1**: Either there is a limitation on the elements in E(D, C) for which the morphism exists. That restriction is catched in the interplay between them and the components of cube A which is described via the previous relation.
- **type 2**: Or there is no limitation on the elements in E(D, C) but the deforming cube A must be such that:

$$A^\lambda_{\chi\mu} . A^\chi_{\alpha\beta} - A^\lambda_{\alpha\gamma} . A^\gamma_{\beta\mu} = 0$$

1.2.5 Linear function

A deformed product (whatever its nature is: tensor, exterior, Lie) is systematically associated with the existence of a trivial decomposition (i) entirely depending on a pair (cube, projectile) and (ii) acting on the left side of the target in the dual representation of E(D, C). This fact allows to consider the trivial decomposition as the image of a linear function ${}_A\Phi$ acting on the elements in E(D, C).

$${}_A\Phi : \mathbf{u} \in E(D,\, C) \;\rightarrow\; {}_A\Phi(\mathbf{u}) \;=\; [A^{\lambda}_{\chi\mu} \,.\, u^{\chi}] \in M(D,\, C)$$

The linearity is easily proved:

$$\mathbf{u},\, \mathbf{v} \in E(D,\, C) :$$

$${}_A\Phi(\mathbf{u} \,+\, \mathbf{v})$$

$$=$$

$$[A^{\lambda}_{\chi\mu} \,.\, (u^{\chi} \,+\, v^{\chi})]$$

$$=$$

$$[A^{\lambda}_{\chi\mu} \,.\, u^{\chi}] \;+\; [A^{\lambda}_{\chi\mu} \,.\, v^{\chi}]$$

$$=$$

$${}_A\Phi(\mathbf{u}) \;+\; {}_A\Phi(\mathbf{v})$$

On the same vein:

$$\forall\, z \in C, \mathbf{u} \in E(D,\, C) :$$

$${}_A\Phi(z \,.\, \mathbf{u}) \;=\; [A^{\lambda}_{\chi\mu} \,.\, z \,.\, u^{\chi}] \;=\; z \,.\, [A^{\lambda}_{\chi\mu} \,.\, u^{\chi}] \;=\; z \,.\, {}_A\Phi(\mathbf{u})$$

Let denote ${}_A\Phi(0)$, the set of all trivial decompositions which can be calculated with the function ${}_A\Phi$ when $\mathbf{u}$ is browsing E(D, C); this set is a subset of M(D, C):

$${}_A\Phi(0) \subset \; M(D,\, C)$$

The function ${}_A\Phi$ itself is an element in L(E(D, C); M(D, C)). It connects two vector spaces.

1.2.6 The functor Φ

At a quite more abstract level, Φ can be interpreted as a functor connecting $\mathrm{C}_{(D-D-D)}$ and L(E(D, C); M(D, C)):

$$A \in C_{(D-D-D)} \to {}_{A}\Phi \in L(E(D,\, C); M(D,\, C))$$

Provided these definitions are accepted:

$$A \;+\; B \equiv A^{\alpha}_{\chi\beta} \;+\; B^{\alpha}_{\chi\beta} \;=\; (A \;+\; B)^{\alpha}_{\chi\beta} \;=\; K^{\alpha}_{\chi\beta} \equiv K$$

$$\forall\, z \in C, z\,.\,A \equiv\; z\,.\,A^{\alpha}_{\chi\beta} \;=\; (z\,.\,A)^{\alpha}_{\chi\beta} \;=\; K^{\alpha}_{\chi\beta} \equiv K$$

this functor is a linear one.

2 Derivations

2.1 Jacobi's territory: existence

Let suppose that such a territory exists or, equivalently, is not empty; then:

$$\otimes_A(\otimes_A(\mathbf{u},\, \mathbf{v}),\, \mathbf{w})$$
$$=$$
$$\otimes_A(A^{\chi}_{\alpha\beta} \,.\, u^{\alpha} \,.\, v^{\beta} \,.\, \mathbf{e}_{\chi},\, w^{\delta} \,.\, \mathbf{e}_{\delta})$$
$$=$$
$$A^{\epsilon}_{\chi\delta} \,.\, A^{\chi}_{\alpha\beta} \,.\, u^{\alpha} \,.\, v^{\beta} \,.\, w^{\delta} \,.\, \mathbf{e}_{\epsilon}$$

And:

$$\otimes_A(\mathbf{v},\, \otimes_A(\mathbf{u},\, \mathbf{w}))$$
$$=$$
$$\otimes_A(v^{\chi} \,.\, \mathbf{e}_{\chi},\, A^{\delta}_{\alpha\beta} \,.\, u^{\alpha} \,.\, w^{\beta} \,.\, \mathbf{e}_{\delta})$$
$$=$$
$$A^{\epsilon}_{\chi\delta} \,.\, v^{\chi} \,.\, A^{\delta}_{\alpha\beta} \,.\, u^{\alpha} \,.\, w^{\beta} \,.\, \mathbf{e}_{\epsilon}$$

And:

$$\otimes_A(\mathbf{u},\, \otimes_A(\mathbf{v},\, \mathbf{w}))$$
$$=$$
$$\otimes_A(u^{\chi} \,.\, \mathbf{e}_{\chi},\, A^{\delta}_{\alpha\beta} \,.\, v^{\alpha} \,.\, w^{\beta} \,.\, \mathbf{e}_{\delta})$$
$$=$$
$$A^{\epsilon}_{\chi\delta} \,.\, u^{\chi} \,.\, A^{\delta}_{\alpha\beta} \,.\, v^{\alpha} \,.\, w^{\beta} \,.\, \mathbf{e}_{\epsilon}$$

And, since Ω is a canonical basis, at components level:

$$\forall \epsilon \in Ind(D):$$

$$A^{\epsilon}_{\chi\delta} \,.\, u^{\chi} \,.\, A^{\delta}_{\alpha\beta} \,.\, v^{\alpha} \,.\, w^{\beta}$$

$$=$$

$$A^{\epsilon}_{\chi\delta} \,.\, A^{\chi}_{\alpha\beta} \,.\, u^{\alpha} \,.\, v^{\beta} \,.\, w^{\delta} + A^{\epsilon}_{\chi\delta} \,.\, v^{\chi} \,.\, A^{\delta}_{\alpha\beta} \,.\, u^{\alpha} \,.\, w^{\beta}$$

But (i) the discussion is developped with elements in C which is a commutative and an associative algebra; (ii) a lot of indices and subscripts are mute. These facts allow several regroupments resulting in:

$$\{A^{\epsilon}_{\alpha\delta} \,.\, A^{\delta}_{\beta\chi} - (A^{\epsilon}_{\delta\chi} \,.\, A^{\delta}_{\alpha\beta} + A^{\epsilon}_{\beta\delta} \,.\, A^{\delta}_{\alpha\chi})\} \,.\, u^{\alpha} \,.\, v^{\beta} \,.\, w^{\chi} = 0$$

There are two types of situations which are compatible with the existence of a Jacobi's territory:

- **type 1**: Either there is a limitation on the elements in E(D, C) generating a Jacobi's territory. That restriction is catched in the interplay between them and the components of cube A which is described via the previous relation.

- **type 2**: Or there is no limitation on the elements in E(D, C) but the deforming cube A must be such that:

$$A^{\epsilon}_{\alpha\delta} \,.\, A^{\delta}_{\beta\chi} - (A^{\epsilon}_{\delta\chi} \,.\, A^{\delta}_{\alpha\beta} + A^{\epsilon}_{\beta\delta} \,.\, A^{\delta}_{\alpha\chi}) = 0$$

2.1.1 Jacobi's territory of type 1:

For now, it can only be remarked that an element in V_D which would generate a Jacobi's territory of type 1, would have components defining a vanishing volumetric form such that:

$$\{A^{\epsilon}_{\alpha\delta} \,.\, A^{\delta}_{\beta\chi} - (A^{\epsilon}_{\delta\chi} \,.\, A^{\delta}_{\alpha\beta} + A^{\epsilon}_{\beta\delta} \,.\, A^{\delta}_{\alpha\chi})\} \,.\, u^{\alpha} \,.\, u^{\beta} \,.\, u^{\chi} = 0$$

2.1.2 Isotropic vectors

Per definition, in that theory, any isotropic vector $\mathbf{u}$ is such that:

$$\otimes_A(\mathbf{u},\, \mathbf{u}) \,=\, \mathbf{0}$$

For a given element $\mathbf{u}$ in V_D there is perhaps a Jacobi's territory ($\mathbf{u}$, $\mathbf{u}$, $\mathbf{u}$) in E^3(D, C) which is described with the the Jacobi's relation:

$$\otimes_A(\mathbf{u},\, \otimes_A(\mathbf{u},\, \mathbf{u})) \,=\, \otimes_A(\otimes_A(\mathbf{u},\, \mathbf{u}),\, \mathbf{u}) \,+\, \otimes_A(\mathbf{u},\, \otimes_A(\mathbf{u},\, \mathbf{u}))$$

If that vector $\mathbf{u}$ is isotropic, then that relation is always true. Any isotropic vector in V_D generates a specific Jacobi's territory ($\mathbf{u}$, $\mathbf{u}$, $\mathbf{u}$) which is of type 1 because it is described via an interplay between the components: those of cube A and the ones of $\mathbf{u}$.

$$\forall\, \chi \,\in\, Ind(D) :\; A^{\chi}_{\alpha\beta} \,.\, u^{\alpha} \,.\, u^{\beta} \,=\, 0$$

In that theory, an isotropic vector can be interpreted as a set of D quadratic forms.

2.1.3 Example

Let consider an element ($\mathbf{u}$, $\mathbf{v}$, $\mathbf{v}$) in E^3(D, C) and let admit *a priori* that it is a Jacobi's territory. Then, per definition:

$$\otimes_A(\mathbf{u},\, \otimes_A(\mathbf{v},\, \mathbf{v})) \,=\, \otimes_A(\otimes_A(\mathbf{u},\, \mathbf{v}),\, \mathbf{v}) \,+\, \otimes_A(\mathbf{v},\, \otimes_A(\mathbf{u},\, \mathbf{v}))$$

If, furthermore, the vector $\mathbf{v}$ is isotropic, then:

$$\mathbf{0} \,=\, \otimes_A(\otimes_A(\mathbf{u},\, \mathbf{v}),\, \mathbf{v}) \,+\, \otimes_A(\mathbf{v},\, \otimes_A(\mathbf{u},\, \mathbf{v}))$$

This condition can only be realized when the deformed tensor product at hand, $\otimes_A$, is anti-symmetric.

2.1.4 Jacobi's territory of type 2:

For this type, all V_D can be a Jacobi's territory, provided the cube A at hand has the ad hoc property. But what can be learn from that relation? Let suppose it is a priori true, then it remains true in inverting two subscripts. Let do it for α and β; both relations are simultaneously true:

$$A^{\epsilon}_{\alpha\delta} . A^{\delta}_{\beta\chi} - (A^{\epsilon}_{\delta\chi} . A^{\delta}_{\alpha\beta} + A^{\epsilon}_{\beta\delta} . A^{\delta}_{\alpha\chi}) = 0$$

$$A^{\epsilon}_{\beta\delta} . A^{\delta}_{\alpha\chi} - (A^{\epsilon}_{\delta\chi} . A^{\delta}_{\beta\alpha} + A^{\epsilon}_{\alpha\delta} . A^{\delta}_{\beta\chi}) = 0$$

Let add them and then get:

$$A^{\epsilon}_{\delta\chi} . (A^{\delta}_{\alpha\beta} + A^{\delta}_{\beta\alpha}) = 0$$

Since vanishing cubes are meaningless for that discussion, that relation tells that any cube which is anti-symmetric on its subscripts is able to define a Jacobi's territory of type 1.

$$A^{\delta}_{\alpha\beta} + A^{\delta}_{\beta\alpha} = 0$$

2.1.5 Equivalence

A deformed tensor product built with an anti-symmetric cube is a deformed Lie product:

$$\forall\,(\mathbf{a}, \mathbf{b}) \in E^2(D,\, K),\, \forall\, A \,:\, A^k_{ij} + A^k_{ji} = 0$$

$$\otimes_A(\mathbf{a}, \mathbf{b})$$

$$=$$

$$\sum_k (\sum_i \sum_j A^k_{ij} . a^i . b^j) . \mathbf{e}_k$$

$$=$$

$$\sum_k (\sum_{i<j} A^k_{ij} . a^i . b^j + \sum_{i=j} A^k_{ij} . a^i . b^j + \sum_{i>j} A^k_{ij} . a^i . b^j) . \mathbf{e}_k$$

$$=$$

$$\sum_k (\sum_{i<j} A^k_{ij} . a^i . b^j + 0 . a^i . b^j - \sum_{i<j} A^k_{ij} . a^j . b^i) . \mathbf{e}_k$$

$$=$$

$$\sum_k (\sum_{i<j} A^k_{ij} . (a^i . b^j - a^j . b^i) . \mathbf{e}_k$$

This is nothing but, the definition of a deformed Lie product:

$$\forall\,(\mathbf{a}, \mathbf{b}) \in E^2(D,\, K) :$$

$$\forall\, A \,:\, A^k_{ij} + A^k_{ji} = 0,\, \otimes_A(\mathbf{a}, \mathbf{b}) = [\mathbf{a}, \mathbf{b}]_A$$

2.1.6 Lie brackets

When V_D is a Jacobi's territory of type 2, then the deformed tensor product with which E(D, C) is equipped is a deformed Lie product and a Lie bracket because the three relations are true:

$$\forall A : A_{ij}^k + A_{ji}^k = 0$$

$$[\mathbf{a}, \mathbf{a}]_A = \mathbf{0}$$

$$[\mathbf{a}, \mathbf{b}]_A + [\mathbf{b}, \mathbf{a}]_A = \mathbf{0}$$

$$[\mathbf{u}, [\mathbf{v}, \mathbf{w}]_A]_A = [[\mathbf{u}, \mathbf{v}]_A, \mathbf{w}]_A + [\mathbf{v}, [\mathbf{u}, \mathbf{w}]_A]_A$$

A vector space V_D which is a Jacobi's territory of type 2 because of the anti-symmetry of A, is nothing but a C-Lie algebra.

2.2 Associativity for the deformed tensor product

Let consider the double products again:

$$\otimes_A(\otimes_A(\mathbf{u}, \mathbf{v}), \mathbf{w}) = A_{\delta\chi}^{\epsilon} . A_{\alpha\beta}^{\delta} . u^{\alpha} . v^{\beta} . w^{\chi} . \mathbf{e}_{\epsilon}$$

and:

$$\otimes_A(\mathbf{u}, \otimes_A(\mathbf{v}, \mathbf{w})) = A_{\alpha\delta}^{\epsilon} . u^{\alpha} . A_{\beta\chi}^{\delta} . v^{\beta} . w^{\chi} . \mathbf{e}_{\epsilon}$$

A deformed tensor product is associative on V_D when:

$$\otimes_A(\otimes_A(\mathbf{u}, \mathbf{v}), \mathbf{w}) = \otimes_A(\mathbf{u}, \otimes_A(\mathbf{v}, \mathbf{w}))$$

or, equivalently, when:

$$A_{\delta\chi}^{\epsilon} . A_{\alpha\beta}^{\delta} . u^{\alpha} . v^{\beta} . w^{\chi} . \mathbf{e}_{\epsilon} = A_{\alpha\delta}^{\epsilon} . u^{\alpha} . A_{\beta\chi}^{\delta} . v^{\beta} . w^{\chi} . \mathbf{e}_{\epsilon}$$

or also when:

$$(A_{\delta\chi}^{\epsilon} . A_{\alpha\beta}^{\delta} - A_{\alpha\delta}^{\epsilon} . A_{\beta\chi}^{\delta}) . u^{\alpha} . v^{\beta} . w^{\chi} = 0$$

Once again, there are two types of situations allowing the associativity:

- **type 1**: Either there is a limitation on the elements in E(D, C) which are associative. That restriction is catched in the interplay between them and the components of cube A which is described via the previous relation.
- **type 2**: Or there is no limitation on the elements in E(D, C) but the deforming cube A must be such that:

$$A^{\epsilon}_{\delta\chi} \,.\, A^{\delta}_{\alpha\beta} \;-\; A^{\epsilon}_{\alpha\delta} \,.\, A^{\delta}_{\beta\chi} \;=\; 0$$

2.2.1 Associativity of type 1

Empty subsection.

2.2.2 Associativity of type 2

Let recall that a multiplicative morphism of type 2 is related to the condition:

$$A^{\lambda}_{\chi\mu} \,.\, A^{\chi}_{\alpha\beta} \;-\; A^{\lambda}_{\alpha\gamma} \,.\, A^{\gamma}_{\beta\mu} \;=\; 0$$

The substitutions $\chi \to \delta$ and $\lambda \to \epsilon$ transforms it into:

$$A^{\epsilon}_{\delta\mu} \,.\, A^{\delta}_{\alpha\beta} \;-\; A^{\epsilon}_{\alpha\gamma} \,.\, A^{\gamma}_{\beta\mu} \;=\; 0$$

The substitutions $\mu \to \chi$ and $\gamma \to \delta$ transforms then it into:

$$A^{\epsilon}_{\delta\chi} \,.\, A^{\delta}_{\alpha\beta} \;-\; A^{\epsilon}_{\alpha\delta} \,.\, A^{\delta}_{\beta\chi} \;=\; 0$$

This formalism is exactly the one of the condition describing an associative deformed tensor product of type 2. This exploration recovers a well-known result: if V_D is equipped with an associative deformed tensor product $\otimes_A$, the function ${}_A\Phi$: E(D, C) $\to$ M(D, C) is a multiplicative morphism between two vector spaces; precisely: V_D and {M(D, C), .}; here, . denotes the product of two matrices.

In particular, if the cube A is compatible with an associativity of type 2 and anti-symmetric (on its subscripts), then ${}_A\Phi$: E(D, C) $\to$ M(D, C) is a multiplicative morphism between two C-Lie algebras; precisely: $\hat{V}_D$ and {M(D, C), [..., ...]}; here, [..., ...] denotes a Lie bracket acting on the elements in M(D, C).

2.3 Matricial derivation.

2.3.1 Intuition introducing the concept.

Let consider an element ($\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$) in E^3(D, C). Let suppose *a priori* that it is a Jacobi's territory. Then, per definition:

$$\otimes_A(\mathbf{u},\, \otimes_A(\mathbf{v},\, \mathbf{w})) \,=\, \otimes_A(\otimes_A(\mathbf{u},\, \mathbf{v}),\, \mathbf{w}) \,+\, \otimes_A(\mathbf{v},\, \otimes_A(\mathbf{u},\, \mathbf{w}))$$

It is neither a scoop nor difficult to recognize a formal analogy with the Leibniz rule. This analogy is obtained in interpreting the symbolism $\otimes_A(\mathbf{u}, ...)$ as the generic representation for a particular kind of derivation; let for example write:

$$\otimes_A(\mathbf{u},\, ...) \,\equiv\, \partial_{\mathbf{u}}$$

In that first intuitive step, the relation characterizing a Jacobi's territory can be rewritten as:

$$\partial_{\mathbf{u}}(\otimes_A(\mathbf{v},\, \mathbf{w})) \,=\, \otimes_A(\partial_{\mathbf{u}}(\mathbf{v}),\, \mathbf{w}) \,+\, \otimes_A(\mathbf{v},\, \partial_{\mathbf{u}}(\mathbf{w}))$$

This rewriting is, stricto sensu, not the historical formulation of Leibniz rule since that rule was concerning the ordinary derivation acting on a product of two multi-variables (also vector) numerical functions (the result is a scalar) f and g.

$$(f.g)' \,=\, f'.g \,+\, f.g'$$

But obviously, by visual extrapolation, the definition of Jacobi's territory preserves the essence of Leibniz rule:

$$' \,\rightarrow\, \partial_{\mathbf{u}} \,\equiv\, \otimes_A(\mathbf{u},\, ...)$$

$$. \,\rightarrow\, \otimes_A$$

$$F(E(D,\, C);\, C) \,\rightarrow\, E(D,\, C) :\, f(\mathbf{u}) \rightarrow\, \mathbf{u}$$

It translates its meaning on new mathematical sets; more precisely:

1. Any trivial decomposition acts on the left side of elements in E(D, C) in playing a role equivalent to an ordinary (event. partial) derivation by respect for one of the variables of which

a numerical function f(..., u^α, ...) depends on.

The difference lies in the fact that that new kind of derivation is done by respect for a pair (A, projectile).

2. The deformed tensor product at hand acts on pairs in E^2(D, C) and represents the translation of a product between two numerical functions.

3. That intuitive and optical extrapolation translates the discussion concerning multi-variables numerical functions in a vector space. In that sense, that intellectual translation may be interpreted as the representation of an inverse function Π^{-1}.

Let suppose that that function Π^{-1} (i) exists and (ii) connects elements in F(E(D, C); C) to elements in E(D, C). Let also suppose that that function is a linear one.

$$\Pi^{-1} : F(E(D,\, C);\, C) \,\rightarrow\, E(D,\, C),\, \Pi^{-1}(f) \rightarrow\, \mathbf{u}$$

$$\forall\, f_1,\, f_2 \,\in\, F(E(D,\, C);\, C) : \Pi^{-1}(f_1 \,+\, f_2) \,=\, \Pi^{-1}(f_1) \,+\, \Pi^{-1}(f_2)$$
$$\forall\, z \,\in\, C,\, f \,\in\, F(E(D,\, C);\, C) : \Pi^{-1}(z \,.\, f) \,=\, z \,.\, \Pi^{-1}(f)$$

Let complete the list of its specific properties with:

$$\forall\, f_1,\, f_2 \,\in\, F(E(D,\, C);\, C) : \Pi^{-1}(f_1 \,.\, f_2) \,=\, \otimes_A(\Pi^{-1}(f_1),\, \Pi^{-1}(f_2))$$

Let now apply these properties to the Leibniz rule and get:

$$\Pi^{-1}((f.g)')$$
$$=$$
$$\Pi^{-1}(f'.g \,+\, f.g')$$
$$=$$
$$\otimes_A(\Pi^{-1}(f'),\, \Pi^{-1}(g)) \,+\, \otimes_A(\Pi^{-1}(f),\, \Pi^{-1}(g'))$$

Let write the correspondances:

$$\Pi^{-1}(f) \,=\, \mathbf{v};\, \Pi^{-1}(f') \,=\, \otimes_A(\mathbf{u},\, \mathbf{v})$$

$$\Pi^{-1}(g) = \mathbf{v};\ \Pi^{-1}(g') = \otimes_A(\mathbf{u},\, \mathbf{w})$$

And let complete it with:

$$\Pi^{-1}((f.g)') = \otimes_A(\mathbf{u},\, \otimes_A(\mathbf{v},\, \mathbf{w}))$$

Let state the coherence of the intuition identifying the relation characterizing Jacobi's territories and the Leibniz rule when:

$$\otimes_A(\mathbf{u},\, \Pi^{-1}(f)) = \Pi^{-1}(f')$$

It can reasonably be suspected that $\otimes_A(\mathbf{u}, \ldots)$ plays the role of a derivation acting on the elements in E(D, C) when the discussion is restricted to that part of V_D for which any element in E^3(D, C) is a Jacobi's territory. This restriction recalls the considerations already exposed in that document. There is *de facto* a one-to-one correspondance between $\otimes_A(\mathbf{u}, \ldots)$ and the matrix ${}_A\Phi(\mathbf{u})$. This fact allows to start a general discussion in which derivations are represented by matrices. I call them: matricial derivations, justifying the title of that mathematical exploration.

2.3.2 Discussion

In general, a derivation symbolizes an evolution, the passage from one state to the next one. It implicitly depends on diverse parameters influencing the variations of the mathematical object under study (position, speed, field, etc.).

The matricial derivation delegates the role normally played by a set of rules (e.g.: (sin x)'$_x$ = cos x) to a set of matrices when the derivation specifically concerns elements in a vector space.

Since the ${}_A\Phi$ are linear functions acting on elements in E(D, C) and represented by an element in M(D, C), it is legitime to ask if all elements in M(D, C) are de facto representing matricial derivations or if the label *matricial derivation* is only awarded when one (or several) supplementary criterion(s) is (are) fulfilled?

The historical concept of (ordinary) derivation is intimely related to the one of limit and describes the increase or the decrease of a numerical value.

$$f'(x_0) \;=\; Lim_{x \,\to\, x_0} \frac{f(x) \;-\; f(x_0)}{x \;-\; x_0}$$

The link between the concept of matricial derivation and the one of limit does not appear here; at least not at a first glance. Any matrix acting on the left side of a vector contains an information transforming that vector and, a priori, that's all.

$$|\mathbf{v} > \rightarrow \; |\mathbf{v} >' = \; [P] \,.\, |\mathbf{v} >$$

This is in peculiar true when the matrix [P] representing the derivation coincides with the trivial decomposition of some deformed tensor product; in that case:

$$|\mathbf{v} > \rightarrow \; |\mathbf{v} >' = \; {}_A\Phi(\mathbf{u}) \,.\, |\mathbf{v} > = \; | \otimes_A (\mathbf{u},\, \mathbf{v}) >$$

In that context, there are yet a lot of meaningful remaining interrogations:

1. What kind of derivation exactly is a matricial derivation? Does it represent the action of a polynomial of degree two (a surface) on a given vector?

2. If this guess is the correct interpretation, are there several pairs ([P], **z**) -each of them symbolizing one polynomial- such that:

   $$|\mathbf{v} > \rightarrow \; |\mathbf{v} >' = \; [P] \,.\, |\mathbf{v} > \; + |\mathbf{z} > = \; | \otimes_A (\mathbf{u},\, \mathbf{v}) > ?$$

   This is the so-called (E) question.

3. If the concept of derivation must stay related to the one of integration, then (i) knowing a given element in E(D, C), (ii) supposing a priori that it is a deformed tensor product of which the second argument (the target) represents the result of an integration of that given element, can we discover (i) the rule discribing and (ii) the actors involved in the derivation?

3 Resumé

This short memoir introduces an intuitive concept of matricial derivation. A matricial derivation delegates the role normally played by a set of rules acting on numerical functions (e.g.: $(\text{sin x})'_x = \text{cos x}$) to a set of matrices acting on elements in a vector space. The deformed tensor products and their diverse decompositions, especially the trivial ones, are the guiding common threads illustrating this concept. The progression explains why this concept justifies to ask the (E) question.