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This document is a reworked version of an old and personal exploration mimicking E. Cartan's work. It is envisaging variations of the local tetrad until the second order and studying their projections. As a first result one recovers a pseudo-Riemann curvature tensor; this is today not a scoop if one identifies the projectors with the local spin connection. As a second result one unexpectedly discovers the existence of fields mimicking the EM fields, except that they are induced by the variations of the geometry.

CONTENT

BASICS

FRAMES

The starting point for the construction of A. Einstein's theory of relativity (GTR; [01]) is in fact an old discussion concerning exchanges of informations between two observers disserting on the same event. That is we have two frames denoted R_1 and R_2 respectively referred to the points O_1 and O_2. The latter are called "origins" and are the respective positions from which a supposed unique phenomenon, denoted by "φ", is studied.

The persons situated in O_1 and in O_2 have implicitly the capacity to follow the phenomenon, which actually means that they are able to attribute a position and a speed to that phenomenon; at each given moment of their respective local chronology (for example two controllers are able to follow a plane because they both own radars).

Each observer is thinking classically. This is resulting in two affirmations: "The observer in O_i says that, at t_i, the phenomenon is localized in position $\varphi_i({}_ix^1, {}_ix^2, {}_ix^3)$ with the speed $u_i({}_iu^1, {}_iu^2, {}_iu^3)$; and i = 1 and 2". We also suppose that O_1 and O_2 can communicate together and exchange informations on that phenomenon.

The first pragmatic problem when two persons want to discuss about a given observation is to be certain that they are really discussing about the same object. This is absolutely not trivial and, because of that, they have to define a concept of basis or of frame. We shall come back later on that important item.

TIME

Within the GTR approach, the time has been introduced. It is a real scalar component placed on an equal foot with the spatial components. Hugged discussions and debates have been made during the last century on what the time exactly is. There are indeed multiple possibilities to dissert about that item. Each of them is in fact depending on the specialist involved in the discussion or on the subjective perception concerning the time. More or less recent discussions are evocating a link between apparently separate concepts in physics: gravitation, time, entropy. Although the debate stays actually totally open and in development, there is perhaps one fact that can unify all opinions: from the human point of view, there is an inexorable evolution from the starting point (the birth) to the end point (the death). This is valid for all objects in our perceptible universe: alive or inert. For particles, this fact is related to the decay phenomenon.

CHARACTERISTICS OF THAT APPROACH

THE ARROW OF TIME

For us it appears to be evidence. All this is giving the sensation that the time is correlated with a kind of directional evolution. This is why we shall suppose –**and this will be the first characteristic of our theory**- that the time is, at least locally, a vector axis.

E. Cartan's legacy

At the beginning of that theory and just seeking for simplicity we shall suppose that O_1 is the origin of a rigid universal 4D frame whilst O_2 is the origin of any 4D frame. This is immediately allowing:

(01)

$$\forall \alpha = 0, 1, 2 \text{ and } 3: \varphi_1 = {}_1x^{\alpha}. \varepsilon_{\alpha}$$

And:

(02)

$$\forall \lambda = 0, 1, 2 \text{ and } 3: \varphi_2 = {}_2x^{\lambda}. e_{\lambda}$$

Let us suppose that an ordinary derivation is defined in the universal frame. We get:

(03)

$${}_1d\varphi_1 = {}_1d_1x^{\alpha}. \varepsilon_{\alpha} + {}_1x^{\alpha}. {}_1d\varepsilon_{\alpha} = {}_1d_1x^{\alpha}. \varepsilon_{\alpha}$$

Indeed, the rigidity of the universal frame implies: ${}_1d\varepsilon_{\alpha} = 0$ (The axes of that frame don't move).

Let us now suppose that the method of the mobile triad initially developed by E. Cartan can be extended to the 4D changing frame (**and this will be the second characteristic of our theory**):

(04)

$${}_2de_{\lambda} = \omega^{\mu}{}_{\lambda}. e_{\mu}$$

This is yielding:

(05)

$${}_2d\varphi_2 = {}_2d_2x^{\lambda}. e_{\lambda} + {}_2x^{\lambda}. {}_2de_{\lambda} = {}_2d_2x^{\lambda}. e_{\lambda} + {}_2x^{\lambda}. \omega^{\mu}{}_{\lambda}. e_{\mu} = ({}_2d_2x^{\lambda}. \delta^{\mu}{}_{\lambda} + {}_2x^{\lambda}. \omega^{\mu}{}_{\lambda}). e_{\mu}$$

Let us suppose that Taylor Mac Laurin's developments of the following formalism are possible:

(06)

$${}_2de_{\lambda} = \frac{\partial e_{\lambda}}{\partial\ {}_2x^{\alpha}}. {}_2d({}_2x^{\alpha}) + ½. \frac{\partial^2 e_{\lambda}}{\partial\ {}_2x^{\beta}\partial\ {}_2x^{\alpha}}. {}_2d({}_2x^{\alpha}). {}_2d({}_2x^{\beta}) + 0(3)$$

And, in order to understand the consequences of that hypothesis, let us accept to believe that the first and the second partial derivates of the local changing tetrad have a projection in the tetrad (**and this will be the third characteristic of our theory**). This is:

(07)

$$\frac{\partial e_{\lambda}}{\partial\ {}_2x^{\alpha}} = P_{\alpha\lambda}{}^{\theta}. e_{\theta}$$

And:

(08)

$$\frac{\partial^2 e_{\lambda}}{\partial\ {}_2x^{\beta}\partial\ {}_2x^{\alpha}} = T_{\beta\alpha\lambda}{}^{\theta}. e_{\theta}$$

Injecting (07) and (08) into (06) yields:

(09)

$${}_2de_{\lambda} = \{P_{\alpha\lambda}{}^{\theta}. {}_2d({}_2x^{\alpha}) + ½. T_{\beta\alpha\lambda}{}^{\theta}. {}_2d({}_2x^{\alpha}). {}_2d({}_2x^{\beta})\}. e_{\theta} + 0(3)$$

This allows a comparison with (04):

(10)

$$\omega^{\theta}{}_{\lambda} = P_{\alpha\lambda}{}^{\theta}. dx^{\alpha} + ½. T_{\beta\alpha\lambda}{}^{\theta}. dx^{\alpha}. dx^{\beta} + \dots$$

(Since there is no ambiguity on the fact that we now work in R_2, we have abandoned the subscript 2). Thus, if the hypothesis of the mobile tetrad has not been done at the beginning of the calculations but if Taylor Mac Laurin's developments are possible, then an element [ω] of $M_4(K)$ can always be defined by approximation and the latter plays an equivalent role than the matrix defining the local transformations of the basis $\Omega(O_2)$ = (... , e_λ, ...) for λ = 0, 1, 2 and 3 via an ordinary derivation d.

THE OBJECTS OF ANHOLONOMICITY

Usually, at this step, some treatments of the GTR introduce the so-called objects of anholonomicity with the purpose to solve (04) or its dual representation. Any conic written in a 4D space can always be decomposed as if it was a conic written in a 3D space. For example, the relation (10) can be seen as a set of 16 such 3D conics:

(11)

$$\omega^\theta{}_\lambda = P_{\alpha\lambda}{}^\theta .\ dx^\alpha + ½.\ T_{\beta\alpha\lambda}{}^\theta .\ dx^\alpha .\ dx^\beta + \ldots$$

$$\downarrow$$

$$- \omega^\theta{}_\lambda + P_{k\lambda}{}^\theta .\ dx^k + P_{0\lambda}{}^\theta .\ dx^0 + ½.\ T_{pk\lambda}{}^\theta .\ dx^k .\ dx^p + ½.\ T_{0k\lambda}{}^\theta .\ dx^k .\ dx^0 + ½.\ T_{k0\lambda}{}^\theta .\ dx^0 .\ dx^k + ½.\ T_{00\lambda}{}^\theta .\ (dx^0)^2 = 0$$

$$\downarrow$$

$$½.\ T_{pk\lambda}{}^\theta .\ dx^k .\ dx^p + \{P_{k\lambda}{}^\theta + ½.\ (T_{0k\lambda}{}^\theta + T_{k0\lambda}{}^\theta).\ dx^0\}.\ dx^k + \{-\omega^\theta{}_\lambda + P_{0\lambda}{}^\theta .\ dx^0 + ½.\ T_{00\lambda}{}^\theta .\ (dx^0)^2\} = 0$$

For each of the 16 conics (λ, θ = 0, 1, 2 and 3):

(12)

$${}_\lambda{}^\theta c_{kp} = ½.\ T_{pk\lambda}{}^\theta$$

(13)

$${}_\lambda{}^\theta c_k = \{P_{k\lambda}{}^\theta + ½.\ (T_{0k\lambda}{}^\theta + T_{k0\lambda}{}^\theta).\ dx^0\}$$

(14)

$${}_\lambda{}^\theta c = -\omega^\theta{}_\lambda + P_{0\lambda}{}^\theta .\ dx^0 + ½.\ T_{00\lambda}{}^\theta .\ (dx^0)^2$$

THE PROBLEMATIC OF THE CONTINUITY

Let us denote each conic with $f^\theta{}_\lambda(\varphi_2)$. We immediately remark that there is a priori no warranty concerning the continuity. The latter is realized if:

(15)

$${}_\lambda{}^\theta c_{kp} = {}_\lambda{}^\theta c_{pk} \Leftrightarrow T_{pk\lambda}{}^\theta = T_{kp\lambda}{}^\theta$$

DEFINING AN INTERVAL OF TIME

All this is in some way naturally introducing the problematic related to the definition of an interval of time. For simplicity we shall accept that only one interval of time can be defined. This is imposing:

(16)

$$(P_{0\lambda}{}^\theta)^2 + 2.\ (\omega^\theta{}_\lambda + {}_\lambda{}^\theta c).\ T_{00\lambda}{}^\theta = 0$$

(17)

$$T_{00\lambda}{}^\theta .\ dx^0 = -\ P_{0\lambda}{}^\theta$$

This is introducing a constraint on the components of the coordinates of $\partial_0 e_\lambda$ and of $\partial^2{}_{00} e_\lambda$:

(18)

$$\partial_0 e_\lambda = P_{0\lambda}{}^{\theta} . e_\theta = - dx^0 . T_{00\lambda}{}^{\theta} . e_\theta = - dx^0 . \partial^2{}_{00} e_\lambda$$

Is that constraint in contradiction with the usual rules of derivation? We expect:

(19)

$$d(\partial_0 e_\lambda) \sim dx^0 . \partial^2{}_{00} e_\lambda + dx^k . \partial^2{}_{k0} e_\lambda$$

Our definition finally yields:

(20)

$$d(\partial_0 e_\lambda) \sim - (\partial_0 e_\lambda) + dx^k . \partial^2{}_{k0} e_\lambda$$

All this first part of the exposé only is an introductory paragraph of the whole theory. Let us suppose (for any obscure reason) that the very classical gradient of $(\partial_0 e_\lambda)$ is in some way orthogonal to the local 3D classical speed vector. Then we get $dx^k . \partial^2{}_{k0} e_\lambda \sim 0$ and we are left with $d(\partial_0 e_\lambda) \sim - (\partial_0 e_\lambda)$. That relation indicates exponentially decreasing variations of the partial derivatives of the basis vectors by respect for the time (axis). This is suggesting that any basis is evolving so that it systematically becomes a rigid basis. It is, at the first glance, difficult to understand the meaning of that relation but, as we shall see later in that document, there is a physically relevant explanation related to the existence of EM fields induced by the variations of the geometry.

The pseudo Riemann curvature tensor

Calculations

The above exposed characteristics of our theory can be generalized. For example the existence of a Taylor Mac Laurin development can be written:

(H-01)

$$de_\lambda = \frac{\partial e_\lambda}{\partial x^{\alpha_1}} . dx^{\alpha_1} + \tfrac{1}{2} . \frac{\partial^2 e_\lambda}{\partial x^{\alpha_2}\partial x^{\alpha_1}} . dx^{\alpha_1} . dx^{\alpha_2} + \dots + \frac{1}{k!} . \frac{\partial^k e_\lambda}{\partial x^{\alpha_k} \dots \partial x^{\alpha_1}} . dx^{\alpha_1} . \dots . dx^{\alpha_k} + \dots + \frac{1}{p!} . \frac{\partial^p e_\lambda}{\partial x^{\alpha_p} \dots \partial x^{\alpha_1}} . dx^{\alpha_1} . \dots . dx^{\alpha_p} + 0(p + 1)$$

The existence of projectors can be written:

(H-02)

$$\forall \lambda = 0, 1, 2, 3 \text{ et } \forall\, k = 1, \dots , p: \frac{\partial^k e_\lambda}{\partial x^{\alpha_k} \dots \partial x^{\alpha_1}} = T_{\lambda \alpha_1 \dots \alpha_k}{}^{\beta} . e_\beta$$

Putting the projectors into the development yields:

(21)

$$de_\lambda = \sum_{k=1}^{k=p} \frac{1}{k!} . T_{\lambda \alpha_1 \dots \alpha_k}{}^{\beta} . dx^{\alpha_1} . \dots . dx^{\alpha_k} . e_\beta$$

And the generalization of the mobile tetrad writes:

(22)

$$[\omega] = [\omega_\lambda{}^{\beta}]$$

$$\omega_\lambda{}^{\beta} = \sum_{k=1}^{k=p} \frac{1}{k!} . T_{\lambda \alpha_1 \dots \alpha_k}{}^{\beta} . dx^{\alpha_1} . \dots . dx^{\alpha_k}$$

From that relation (22) one can always define a cube $\blacktriangledown\omega$ such that:

(23)

$$\omega_\lambda{}^{\beta} = \sum_{k=1}^{k=p} \frac{1}{k!} . T_{\lambda \alpha_1 \dots \alpha_k}{}^{\beta} . dx^{\alpha_1} . \dots . dx^{\alpha_k} = \omega_\lambda{}^{\beta}{}_{\alpha_1} . dx^{\alpha_1}$$

with the help of the following relation:

(24)

$$\omega_{\lambda\ \alpha_1}^{\ \beta} = T_{\lambda\alpha_1}^{\ \ \beta} + ½.\ T_{\lambda\alpha_1\alpha_2}^{\ \ \ \beta}.\ dx^{\alpha_2} + \ldots$$

Since the relation (24) must be true for any indice:

(24-A)

$$\omega_{\lambda\ \alpha_2}^{\ \beta} = T_{\lambda\alpha_2}^{\ \ \beta} + ½.\ T_{\lambda\alpha_2\alpha_1}^{\ \ \ \beta}.\ dx^{\alpha_1} + \ldots$$

And:

(24-B)

$$(½.\ T_{\lambda\alpha_1\alpha_2}^{\ \ \ \beta}.\ dx^{\alpha_1}.\ dx^{\alpha_2} + \ldots) - (½.\ T_{\lambda\alpha_2\alpha_1}^{\ \ \ \beta}.\ dx^{\alpha_1}.\ dx^{\alpha_2} + \ldots) = (\omega_{\lambda\ \alpha_1}^{\ \beta} - T_{\lambda\alpha_1}^{\ \ \beta}).\ dx^{\alpha_1} - (\omega_{\lambda\ \alpha_2}^{\ \beta} - T_{\lambda\alpha_2}^{\ \ \beta}).\ dx^{\alpha_2}$$

Or:

$$½.\ (T_{\lambda\alpha_1\alpha_2}^{\ \ \ \beta} - T_{\lambda\alpha_2\alpha_1}^{\ \ \ \beta}).\ dx^{\alpha_1}.\ dx^{\alpha_2} + \ldots = (\omega_{\lambda\ \alpha_1}^{\ \beta} - T_{\lambda\alpha_1}^{\ \ \beta}).\ dx^{\alpha_1} - (\omega_{\lambda\ \alpha_2}^{\ \beta} - T_{\lambda\alpha_2}^{\ \ \beta}).\ dx^{\alpha_2}$$

Let us consider the relations (07) and (08) again and let us calculate the first partial derivatives of (07):

(25)

$$\frac{\partial^2 e_\lambda}{\partial x^\theta \partial x^\alpha} = \partial_\theta T_{\lambda\alpha}^{\ \ \beta}.\ e_\beta + T_{\lambda\alpha}^{\ \ \beta}.\ \partial_\theta e_\beta = \partial_\theta T_{\lambda\alpha}^{\ \ \mu}.\ e_\mu + T_{\lambda\alpha}^{\ \ \beta}.\ T_{\beta\theta}^{\ \ \mu}.\ e_\mu = (\partial_\theta T_{\lambda\alpha}^{\ \ \mu} + T_{\lambda\alpha}^{\ \ \beta}.\ T_{\beta\theta}^{\ \ \mu}).\ e_\mu$$

A comparison with (08) yields:

(26)

$$T_{\lambda\alpha\theta}^{\ \ \ \mu} = \partial_\theta T_{\lambda\alpha}^{\ \ \mu} + T_{\lambda\alpha}^{\ \ \beta}.\ T_{\beta\theta}^{\ \ \mu}$$

Since the relation (26) must be true for any indicia:

(27)

$$T_{\lambda\theta\alpha}^{\ \ \ \mu} = \partial_\alpha T_{\lambda\theta}^{\ \ \mu} + T_{\lambda\theta}^{\ \ \beta}.\ T_{\beta\alpha}^{\ \ \mu}$$

From which we deduce:

(28)

$$\frac{\partial^2 e_\lambda}{\partial x^\theta \partial x^\alpha} - \frac{\partial^2 e_\lambda}{\partial x^\alpha \partial x^\theta} = (T_{\lambda\alpha\theta}^{\ \ \ \mu} - T_{\lambda\theta\alpha}^{\ \ \ \mu}).\ e_\mu = (\partial_\theta T_{\lambda\alpha}^{\ \ \mu} - \partial_\alpha T_{\lambda\theta}^{\ \ \mu} + T_{\lambda\alpha}^{\ \ \beta}.\ T_{\beta\theta}^{\ \ \mu} - T_{\lambda\theta}^{\ \ \beta}.\ T_{\beta\alpha}^{\ \ \mu}).\ e_\mu$$

It is known that, within the "classical" GTR approach [01], we have the following relations:

(29)

$$R_{\lambda\ \theta\alpha}^{\ \mu} = \partial_\theta \Gamma_{\lambda\alpha}^{\ \ \mu} - \partial_\alpha \Gamma_{\lambda\theta}^{\ \ \mu} + \Gamma_{\lambda\alpha}^{\ \ \beta}.\ \Gamma_{\beta\theta}^{\ \ \mu} - \Gamma_{\lambda\theta}^{\ \ \beta}.\ \Gamma_{\beta\alpha}^{\ \ \mu}$$

Our result is thus suggesting the definition of some pseudo Riemannian tensor with the same formalism:

$$_{pseudo}R_{\lambda\ \theta\alpha}^{\ \mu} = \partial_\theta T_{\lambda\alpha}^{\ \ \mu} - \partial_\alpha T_{\lambda\theta}^{\ \ \mu} + T_{\lambda\alpha}^{\ \ \beta}.\ T_{\beta\theta}^{\ \ \mu} - T_{\lambda\theta}^{\ \ \beta}.\ R_{\beta\alpha}^{\ \ \mu}$$

That definition yields:

$$\frac{\partial^2 e_\lambda}{\partial x^\theta \partial x^\alpha} - \frac{\partial^2 e_\lambda}{\partial x^\alpha \partial x^\theta} = {}_{pseudo}R_{\lambda\ \theta\alpha}^{\ \mu}.\ e_\mu$$

This is suggesting the necessity of a comparison between the "classical" GTR approach (no torsion, symmetric metric and so and) and the E. Cartan's approach. Evidently, in case of equality between the Christoffel's cube and the cube ▼T of the E. Cartan approach, the relation (24-B) allows:

(30)

$$½.\ R_{\lambda\alpha_1\alpha_2}{}^{\beta}.\ dx^{\alpha_1}.\ dx^{\alpha_2} + \ldots = (\omega_{\lambda}{}^{\beta}{}_{\alpha_1} - T_{\lambda\alpha_1}{}^{\beta}).\ dx^{\alpha_1} - (\omega_{\lambda}{}^{\beta}{}_{\alpha_2} - T_{\lambda\alpha_2}{}^{\beta}).\ dx^{\alpha_2}$$

Referring to the "classical" GTR literature the above l.h.t is representing components of a mixed tensor:

(31)

$$½.\ \Omega_{\lambda}{}^{\beta} + \ldots = (\omega_{\lambda}{}^{\beta}{}_{\alpha_1} - T_{\lambda\alpha_1}{}^{\beta}).\ dx^{\alpha_1} - (\omega_{\lambda}{}^{\beta}{}_{\alpha_2} - T_{\lambda\alpha_2}{}^{\beta}).\ dx^{\alpha_2}$$

That tensor vanishes if terms of the second order are negligible. One must notice that the equality ▼T = ▼Γ is far to be the unique solution insuring an identity between the two different approaches of the Riemann Christoffel tensor. But we leave the development of that point for later.

THE PRICE FOR NON LINEARITY

Let us suppose that $E_4(K)$ is equipped with a bilinear function denoted $<\ldots, \ldots>_G$: $E_4(K) \times E_4(K) \rightarrow K$. And let us suppose that the local metric is defined with the help of what we call a Pythagoras table (a generic table of multiplication). We write:

(32)

$$T_2(<\ldots, \ldots>)(\Omega, \Omega) = G$$

Since we have already supposed that the ordinary derivation is defined:

(33)

$$dg_{\lambda\mu} = < de_{\lambda}, e_{\mu} >_G + < e_{\lambda}, de_{\mu} >_G$$

With the help of (21) and (22):

(34)

$$dg_{\lambda\mu} = < \omega_{\lambda}{}^{\beta}.\ e_{\beta}, e_{\mu} >_G + < e_{\lambda}, \omega_{\mu}{}^{\beta}.\ e_{\beta} >_G = \omega_{\lambda}{}^{\beta}.< e_{\beta}, e_{\mu} >_G + \omega_{\mu}{}^{\beta}.< e_{\lambda}, e_{\beta} >_G = \omega_{\lambda}{}^{\beta}.\ g_{\beta\mu} + \omega_{\mu}{}^{\beta}.\ g_{\lambda\beta}$$

After that:

(35)

$$dg_{\lambda\mu} = \{\sum_{k=1}^{k=p} \frac{1}{k!}.T_{\lambda\alpha_1\ldots\alpha_k}{}^{\beta}.\ dx^{\alpha_1}.\ \ldots\ .\ dx^{\alpha_k}\}.\ g_{\beta\mu} + \{\sum_{k=1}^{k=p} \frac{1}{k!}.T_{\mu\alpha_1\ldots\alpha_k}{}^{\beta}.\ dx^{\alpha_1}.\ \ldots\ .\ dx^{\alpha_k}\}.\ g_{\lambda\beta}$$

This can be reorganized:

(36)

$$dg_{\lambda\mu} = \sum_{k=1}^{k=p} \frac{1}{k!}.(T_{\lambda\alpha_1\ldots\alpha_k}{}^{\beta}.\ g_{\beta\mu} + T_{\mu\alpha_1\ldots\alpha_k}{}^{\beta}.\ g_{\lambda\beta}).\ dx^{\alpha_1}.\ \ldots\ .\ dx^{\alpha_k}$$

And this is suggesting:

(37)

$$\frac{\partial^k g_{\lambda\mu}}{\partial x^{\alpha_k}\ldots\partial x^{\alpha_1}} = T_{\lambda\alpha_1\ldots\alpha_k}{}^{\beta}.\ g_{\beta\mu} + T_{\mu\alpha_1\ldots\alpha_k}{}^{\beta}.\ g_{\lambda\beta}$$

Let us then consider the k – 1 th partial derivates and derivate them once more time:

(38)

$$\frac{\partial}{\partial x^{\alpha_k}}(\frac{\partial^{k-1} g_{\lambda\mu}}{\partial x^{\alpha_{k-1}}\ldots\partial x^{\alpha_1}})$$

$$= \frac{\partial}{\partial x^{\alpha_k}}(T_{\lambda\alpha_1 \ldots \alpha_{k-1}}{}^{\beta} . g_{\beta\mu} + T_{\mu\alpha_1 \ldots \alpha_{k-1}}{}^{\beta} . g_{\lambda\beta}) = \frac{\partial}{\partial x^{\alpha_k}} T_{\lambda\alpha_1 \ldots \alpha_{k-1}}{}^{\beta} . g_{\beta\mu} + T_{\lambda\alpha_1 \ldots \alpha_{k-1}}{}^{\beta} . \frac{\partial}{\partial x^{\alpha_k}} g_{\beta\mu} + \frac{\partial}{\partial x^{\alpha_k}} T_{\mu\alpha_1 \ldots \alpha_{k-1}}{}^{\beta} . g_{\lambda\beta} + T_{\mu\alpha_1 \ldots \alpha_{k-1}}{}^{\beta} . \frac{\partial}{\partial x^{\alpha_k}} g_{\lambda\beta}$$

With the help of (37):
(39)

$$\frac{\partial}{\partial x^{\alpha_k}} g_{\beta\mu} = T_{\beta\alpha_k}{}^{\theta} . g_{\theta\mu} + T_{\mu\alpha_k}{}^{\theta} . g_{\beta\theta}$$

$$\frac{\partial}{\partial x^{\alpha_k}} g_{\lambda\beta} = T_{\lambda\alpha_k}{}^{\theta} . g_{\theta\beta} + T_{\beta\alpha_k}{}^{\theta} . g_{\lambda\theta}$$

Introducing it into (38):
(40)

$$\frac{\partial}{\partial x^{\alpha_k}}\left(\frac{\partial^{k-1} g_{\lambda\mu}}{\partial x^{\alpha_{k-1}} \ldots \partial x^{\alpha_1}}\right) = \frac{\partial}{\partial x^{\alpha_k}} T_{\lambda\alpha_1 \ldots \alpha_{k-1}}{}^{\beta} . g_{\beta\mu} + T_{\lambda\alpha_1 \ldots \alpha_{k-1}}{}^{\beta} . (T_{\beta\alpha_k}{}^{\theta} . g_{\theta\mu} + T_{\mu\alpha_k}{}^{\theta} . g_{\beta\theta}) + \frac{\partial}{\partial x^{\alpha_k}} T_{\mu\alpha_1 \ldots \alpha_{k-1}}{}^{\beta} . g_{\lambda\beta} + T_{\mu\alpha_1 \ldots \alpha_{k-1}}{}^{\beta} . (T_{\lambda\alpha_k}{}^{\theta} . g_{\theta\beta} + T_{\beta\alpha_k}{}^{\theta} . g_{\lambda\theta})$$

Now starting with (H-02) for the k – 1 th order:
(41)

$$\frac{\partial^{k-1} e_{\lambda}}{\partial x^{\alpha_{k-1}} \ldots \partial x^{\alpha_1}} = T_{\lambda\alpha_1 \ldots \alpha_{k-1}}{}^{\beta} . e_{\beta}$$

Calculating a partial derivation:
(42)

$$\frac{\partial}{\partial x^{\alpha_k}} \frac{\partial^{k-1} e_{\lambda}}{\partial x^{\alpha_{k-1}} \ldots \partial x^{\alpha_1}} = \frac{\partial}{\partial x^{\alpha_k}}(T_{\lambda\alpha_1 \ldots \alpha_{k-1}}{}^{\beta} . e_{\beta}) = \frac{\partial}{\partial x^{\alpha_k}} T_{\lambda\alpha_1 \ldots \alpha_{k-1}}{}^{\beta} . e_{\beta} + T_{\lambda\alpha_1 \ldots \alpha_{k-1}}{}^{\beta} . \frac{\partial}{\partial x^{\alpha_k}} e_{\beta}$$

And involving (H-02) again at the first order:
(43)

$$\frac{\partial}{\partial x^{\alpha_k}} \frac{\partial^{k-1} e_{\lambda}}{\partial x^{\alpha_{k-1}} \ldots \partial x^{\alpha_1}} = \frac{\partial}{\partial x^{\alpha_k}} T_{\lambda\alpha_1 \ldots \alpha_{k-1}}{}^{\beta} . e_{\beta} + T_{\lambda\alpha_1 \ldots \alpha_{k-1}}{}^{\beta} . T_{\beta\alpha_k}{}^{\theta} . e_{\theta} = \left(\frac{\partial}{\partial x^{\alpha_k}} T_{\lambda\alpha_1 \ldots \alpha_{k-1}}{}^{\theta} + T_{\lambda\alpha_1 \ldots \alpha_{k-1}}{}^{\beta} . T_{\beta\alpha_k}{}^{\theta}\right) . e_{\theta}$$

The (H-02) for the k th yields:

$$\frac{\partial^{k} e_{\lambda}}{\partial x^{\alpha_k} \ldots \partial x^{\alpha_1}} = T_{\lambda\alpha_1 \ldots \alpha_k}{}^{\theta} . e_{\theta}$$

This is finally yielding a very general relation:
(44)

$$T_{\lambda\alpha_1 \ldots \alpha_k}{}^{\theta} = \frac{\partial}{\partial x^{\alpha_k}} T_{\lambda\alpha_1 \ldots \alpha_{k-1}}{}^{\theta} + T_{\lambda\alpha_1 \ldots \alpha_{k-1}}{}^{\beta} . T_{\beta\alpha_k}{}^{\theta}$$

Coming back now to (40):
(45)

$$\frac{\partial}{\partial x^{\alpha_k}}\left(\frac{\partial^{k-1} g_{\lambda\mu}}{\partial x^{\alpha_{k-1}} \dots \partial x^{\alpha_1}}\right)$$
$$=$$
$$\frac{\partial}{\partial x^{\alpha_k}} T_{\lambda\alpha_1 \dots \alpha_{k-1}}{}^{\theta} . g_{\theta\mu} + T_{\lambda\alpha_1 \dots \alpha_{k-1}}{}^{\beta} . (T_{\beta\alpha_k}{}^{\theta} . g_{\theta\mu} + T_{\mu\alpha_k}{}^{\theta} . g_{\beta\theta})$$
$$+$$
$$\frac{\partial}{\partial x^{\alpha_k}} T_{\mu\alpha_1 \dots \alpha_{k-1}}{}^{\theta} . g_{\lambda\theta} + T_{\mu\alpha_1 \dots \alpha_{k-1}}{}^{\beta} . (T_{\lambda\alpha_k}{}^{\theta} . g_{\theta\beta} + T_{\beta\alpha_k}{}^{\theta} . g_{\lambda\theta})$$
$$=$$
$$\left(\frac{\partial}{\partial x^{\alpha_k}} T_{\lambda\alpha_1 \dots \alpha_{k-1}}{}^{\theta} + T_{\lambda\alpha_1 \dots \alpha_{k-1}}{}^{\beta} . T_{\beta\alpha_k}{}^{\theta}\right) . g_{\theta\mu} + T_{\lambda\alpha_1 \dots \alpha_{k-1}}{}^{\beta} . T_{\mu\alpha_k}{}^{\theta} . g_{\beta\theta}$$
$$+$$
$$\left(\frac{\partial}{\partial x^{\alpha_k}} T_{\mu\alpha_1 \dots \alpha_{k-1}}{}^{\theta} + T_{\mu\alpha_1 \dots \alpha_{k-1}}{}^{\beta} . T_{\beta\alpha_k}{}^{\theta}\right) . g_{\lambda\theta} + T_{\mu\alpha_1 \dots \alpha_{k-1}}{}^{\beta} . T_{\lambda\alpha_k}{}^{\theta} . g_{\theta\beta}$$
$$=$$
$$T_{\lambda\alpha_1 \dots \alpha_k}{}^{\theta} . g_{\theta\mu} + T_{\lambda\alpha_1 \dots \alpha_{k-1}}{}^{\beta} . T_{\mu\alpha_k}{}^{\theta} . g_{\beta\theta}$$
$$+$$
$$T_{\mu\alpha_1 \dots \alpha_k}{}^{\theta} . g_{\lambda\theta} + T_{\mu\alpha_1 \dots \alpha_{k-1}}{}^{\beta} . T_{\lambda\alpha_k}{}^{\theta} . g_{\theta\beta}$$

Comparing with (37):

$$\frac{\partial^k g_{\lambda\mu}}{\partial x^{\alpha_k} \dots \partial x^{\alpha_1}}$$
$$=$$
$$T_{\lambda\alpha_1 \dots \alpha_k}{}^{\theta} . g_{\theta\mu} + T_{\mu\alpha_1 \dots \alpha_k}{}^{\theta} . g_{\lambda\theta}$$
$$=$$
$$T_{\lambda\alpha_1 \dots \alpha_k}{}^{\theta} . g_{\theta\mu} + T_{\lambda\alpha_1 \dots \alpha_{k-1}}{}^{\beta} . T_{\mu\alpha_k}{}^{\theta} . g_{\beta\theta}$$
$$+$$
$$T_{\mu\alpha_1 \dots \alpha_k}{}^{\theta} . g_{\lambda\theta} + T_{\mu\alpha_1 \dots \alpha_{k-1}}{}^{\beta} . T_{\lambda\alpha_k}{}^{\theta} . g_{\theta\beta}$$

This is yielding the most important relation of coherence of this theory:

(46)

$$T_{\lambda\alpha_1 \dots \alpha_{k-1}}{}^{\beta} . T_{\mu\alpha_k}{}^{\theta} . g_{\beta\theta} + T_{\mu\alpha_1 \dots \alpha_{k-1}}{}^{\beta} . T_{\lambda\alpha_k}{}^{\theta} . g_{\theta\beta} = 0$$

Indeed, this is also:

(47)

$$T_{\lambda\alpha_1 \dots \alpha_{k-1}}{}^{\beta} . T_{\mu\alpha_k}{}^{\theta} . < e_\beta, e_\theta >_G + T_{\mu\alpha_1 \dots \alpha_{k-1}}{}^{\beta} . T_{\lambda\alpha_k}{}^{\theta} . < e_\theta, e_\beta >_G = 0$$

Or, because (a) K is supposed to be a commutative set and (b) is a bilinear application:

(48)

$$< T_{\lambda\alpha_1 \dots \alpha_{k-1}}{}^{\beta} . e_\beta, T_{\mu\alpha_k}{}^{\theta} . e_\theta >_G + < T_{\lambda\alpha_k}{}^{\theta} . e_\theta, T_{\mu\alpha_1 \dots \alpha_{k-1}}{}^{\beta} . e_\beta >_G = 0$$

At the end and with the help of (H-02) again:

(49)

$$< \frac{\partial^{k-1} e_\lambda}{\partial x^{\alpha_{k-1}} \dots \partial x^{\alpha_1}} , \partial_{\alpha_k} e_\mu >_G + < \partial_{\alpha_k} e_\lambda , \frac{\partial^{k-1} e_\mu}{\partial x^{\alpha_{k-1}} \dots \partial x^{\alpha_1}} >_G = 0$$

This is then yielding the following series:

(50_K)

$$dx^{\alpha_1} . \dots . dx^{\alpha_k} . \left(< \frac{\partial^{k-1} e_\lambda}{\partial x^{\alpha_{k-1}} \dots \partial x^{\alpha_1}} , \partial_{\alpha_k} e_\mu >_G + < \partial_{\alpha_k} e_\lambda , \frac{\partial^{k-1} e_\mu}{\partial x^{\alpha_{k-1}} \dots \partial x^{\alpha_1}} >_G\right) = 0$$
$$\downarrow$$
$$dx^{\alpha_1} . \dots . dx^{\alpha_k} . < \frac{\partial^{k-1} e_\lambda}{\partial x^{\alpha_{k-1}} \dots \partial x^{\alpha_1}} , \partial_{\alpha_k} e_\mu >_G + dx^{\alpha_1} . \dots . dx^{\alpha_k} . < \partial_{\alpha_k} e_\lambda , \frac{\partial^{k-1} e_\mu}{\partial x^{\alpha_{k-1}} \dots \partial x^{\alpha_1}} >_G = 0$$
$$\downarrow$$

$$< \frac{\partial^{k-1} e_\lambda}{\partial x^{\alpha_{k-1}} \dots \partial x^{\alpha_1}} . \mathrm{d}x^{\alpha_1} . \dots . \mathrm{d}x^{\alpha_{k-1}} , \partial_{\alpha_k} \mathrm{e}_\mu . \mathrm{d}x^{\alpha_k} >_G + < \partial_{\alpha_k} \mathrm{e}_\lambda . \mathrm{d}x^{\alpha_k} , \frac{\partial^{k-1} e_\mu}{\partial x^{\alpha_{k-1}} \dots \partial x^{\alpha_1}} . \mathrm{d}x^{\alpha_1} . \dots . \mathrm{d}x^{\alpha_{k-1}} >_G = 0$$

This was for the passage from the k – 1 th to the k th order. Similarly for the passage from the k – 2 th to the k – 1 order:

(50_K – 1)

$$< \frac{\partial^{k-2} e_\lambda}{\partial x^{\alpha_{k-2}} \dots \partial x^{\alpha_1}} . \mathrm{d}x^{\alpha_1} . \dots . \mathrm{d}x^{\alpha_{k-2}} , \partial_{\alpha_{k-1}} \mathrm{e}_\mu . \mathrm{d}x^{\alpha_{k-1}} >_G + < \partial_{\alpha_{k-1}} \mathrm{e}_\lambda . \mathrm{d}x^{\alpha_{k-1}} , \frac{\partial^{k-2} e_\mu}{\partial x^{\alpha_{k-2}} \dots \partial x^{\alpha_1}} . \mathrm{d}x^{\alpha_1} . \dots . \mathrm{d}x^{\alpha_{k-2}} >_G = 0$$

And so and until the passage from the 2th to the 3th order:

(50_3)

$$< \frac{\partial^2 e_\lambda}{\partial x^{\alpha_2} \partial x^{\alpha_1}} . \mathrm{d}x^{\alpha_1} . \mathrm{d}x^{\alpha_2} , \partial_{\alpha_3} \mathrm{e}_\mu . \mathrm{d}x^{\alpha_3} >_G + < \partial_{\alpha_3} \mathrm{e}_\lambda . \mathrm{d}x^{\alpha_3} , \frac{\partial^2 e_\mu}{\partial x^{\alpha_2} \partial x^{\alpha_1}} . \mathrm{d}x^{\alpha_1} . \mathrm{d}x^{\alpha_2} >_G = 0$$

And from the first to the second order:

(50_2)

$$< \frac{\partial \ \ e_\lambda}{\partial x^{\alpha_1}} . \mathrm{d}x^{\alpha_1} , \partial_{\alpha_2} \mathrm{e}_\mu . \mathrm{d}x^{\alpha_2} >_G + < \partial_{\alpha_2} \mathrm{e}_\lambda . \mathrm{d}x^{\alpha_2} , \frac{\partial \ \ e_\mu}{\partial x^{\alpha_1}} . \mathrm{d}x^{\alpha_1} >_G = 0$$

Pseudo-EM fields induced by variations of the geometry

The latter is a very important relation within that approach since it indicates the **natural existence of spinors** [02] in E. Cartan's approach of the generalized theory of relativity. The properties of spinors suggest the existence of fields induced by the variations of the tetrad and isomorphic to EM fields:

(51)

$$\forall\ \alpha, \lambda, \mu = 0, 1, 2, 3 : F_{\lambda\mu} = t^\alpha . <\partial_\lambda e_\alpha, \partial_\mu e_\alpha>_G = t^\alpha . g_{\varepsilon\theta} . T_{\alpha\lambda}{}^\varepsilon . T_{\alpha\mu}{}^\theta$$

Conclusions

Statements

In conclusion, we state that:

1°) The continuity of the presupposed existing conics $f^\theta{}_\lambda(\varphi_2)$ results in the vanishing of a part of the components of the new obtained pseudo Riemann tensor, namely the ${}_{\mathrm{pseudo}}R_p{}^\theta{}_{k\lambda}$.
Because of the Gauss-Codazzi relations [03] this leaves us sometimes only with the components of the extrinsic curvature tensor.

2°) there are deformations of the tetrad generating EM fields.

History

This document is a personal reworking of E. Cartan's methods. It has envisaged variations of the local tetrad until the second order and studied their projections. As a first result it recovers a pseudo-Riemann curvature tensor; this is actually not a scoop if one identifies the projectors ▼T of our approach with E. Cartan's local spin connection (usually denoted ▼ω in the modern literature). As a second result one unexpectedly discovers the existence of fields mimicking the EM fields, except that they are induced by the variations of the geometry.

I wrote the first version of that document at the end of the seventies, ignoring everything on E. Cartan's work, especially on spinors [02]. This is the reason why I have left it in a corner of my room during about thirty years.

LINK WITH THE LORENTZ-EINSTEIN LAW OF MOTION

It actually represents some advantage if one confronts it with another part of my work; namely: the one studying the representations of the Lorentz-Einstein Law (LEL) of motion under a second order self-adjoint differential operator formalism [04]. The reason for the enthusiasm lies in the fact that that formalism allows the appearance of a connection concerning the (up, down) tensor representations of the EM fields. The connecting (4-4) matrices are elements, [G], respecting the so-called "golden rule: The transposed of the inverse is equal to the inverse of the transposed".

$$\{[G]^{-1}\}^{t} = \{[G]^{t}\}^{-1}$$

By side and of interest here, when the initial EM fields of that connection vanish, that connection is characterized by a specific term with the following generic formalism:

$$[F^{\alpha}{}_{\beta}] = [G]^{-1}.\ \delta G$$

This suggests the existence of EM fields resulting from variations of these "golden elements" [G]. Consequently, at this stage, if some local representations of the metric are golden elements, then there are EM-fields such that:

$$[F_{\alpha\beta}] = \delta G$$

This is exactly what the document here above explains and predicts.

My enthusiasm reaches its paroxysm when I state that a third approach of mine, also involving the LEL, yields exactly the same kind of formal result; see [05], [06].

CONCLUDING REMARK

At the end of the day, I would say that converging theoretical proofs concerning the existence of "geometrical EM fields" have been accumulated.

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