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Contents

english

1 Lie, Cartan and Cie.

1.1 The starting point

In the first part, I have explained that any anti-symmetric cube ($A^{\chi}{}_{\alpha\beta} + A^{\chi}{}_{\beta\alpha} = 0 \iff A \in C^{-}{}_{(D-D-D)}$) automatically validates the Jacobi's relation for any element ($\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$) in $V^3{}_D$, transforms the deformed tensor product $\otimes_A$ at hand into a deformed Lie product $[..., ...]_A$ and provides V_D with a C-Lie algebra structure denoted $\hat{V}_D$.

In that first part, Chapter 02, Section 2.3, I have also explained that $\otimes_A(\mathbf{u}, ...)$ plays the role of a derivation acting on the elements in E(D, C) when the discussion is restricted to that part of V_D = {E(D, C), $\otimes_A$} for which any element in E^3(D, C) is a Jacobi's territory.

$$\forall\,(A,\,\mathbf{u}) \in C^{-}_{(D-D-D)} \times E(D,\,C) : [\mathbf{u},\,...]_A \in Der(\hat{V}_D)$$

[1]: Der($\hat{V}_D$) is a C-Lie subalgebra of End($\hat{V}_D$). Per construction, to each given $\mathbf{u}$ in $\hat{V}_D$ corresponds $[\mathbf{u}, ...]_A$ in Der($\hat{V}_D$). This fact authorizes the writing $[\mathbf{u}, ...]_A$ = $ad_{\mathbf{u}}$ insuring the coincidence with usual notations but, only when the discussion stays in $\hat{V}_D$; I mean, only if the cube A which is involved in the discussion does not vary and can, because of that, be ignored. Otherwise, the symbolism must be more explicit; in that case, the best convention perhaps simply is $[\mathbf{u}, ...]_A$.

1.2 Lie-algebras morphism and intern derivation.

A legitime questioning arises from this approach: "Is it just an exercise around the thematic Lie-algebras; the novelty being entirely contained in the deformation of the tensor/Lie product. In which case, general results should be transposable without too much complications? Or is the deformation already the starting point of an extrapolation that could bring new insights on the theory of Lie algebras?" Let illustrate this interrogation in trying to extrapolate the formal relation:

$$ad_{[\mathbf{u},\,\mathbf{v}]} \equiv [ad_{\mathbf{u}},\,ad_{\mathbf{v}}]$$

[1] General results.

Let analyze it attentively and rigorously. The brain interprets "ad" as a function:

$$\mathbf{u} \rightarrow ad(\mathbf{u}) = ad_{[\mathbf{u}]}$$

$$\mathbf{v} \rightarrow ad(\mathbf{v}) = ad_{[\mathbf{v}]}$$

and, following the same logic, gets:

$$[\mathbf{u}, \mathbf{v}] \rightarrow ad([\mathbf{u}, \mathbf{v}]) = ad_{[\mathbf{u}, \mathbf{v}]}$$

Within classical approaches or textbooks, the relation under analyze is supposed to characterize a morphism connecting the sets $\{E(D, C), [..., ...]\}$ and $Der(\{E(D, C), [..., ...]\}) = \{[\mathbf{u}, ...] = ad_{\mathbf{u}}, \mathbf{u} \in E(D, C)\}$.
Nobody contests its effective realization because our brain intuitively supposes that a bracket that has been abusively denoted with the same symbolism [..., ...] than the one which is acting on the elements in {E(D, C), [..., ...]} is also acting on the elements in Der({E(D, C), [..., ...]}).
This is due to the fact that we believe that the bracket acting on elements in Der({E(D, C), [..., ...]}) exists; without proof and without having a precize idea on how it works.
Hence, the relation should be written with more precision (for example in involving two different colors):

$$ad([\mathbf{u}, \mathbf{v}]) = ad_{[\mathbf{u}, \mathbf{v}]} \equiv [ad_{\mathbf{u}}, ad_{\mathbf{v}}] = [ad(\mathbf{u}), ad(\mathbf{v})]$$

That writing would better translate the fact that the function ad is a morphism relating {E(D, C), [..., ...]} to Der({E(D, C), [..., ...]}). Now, even with that precision, we stay with an equivalence. At this stage, nothing has been said on how the red brackets work. A concrete representation for $ad(\mathbf{u})$ is needed. It is got via the following schema:

$$... \rightarrow ad(\mathbf{u})(...) = ad_{\mathbf{u}}(...) = [\mathbf{u}, ...]$$

Following this step by step progression, the morphism is usually represented with the relation:

$$ad([\mathbf{u}, \mathbf{v}])(...) = [ad(\mathbf{u}), ad(\mathbf{v})](...) = [ad(\mathbf{u})(...), ad(\mathbf{v})(...)]$$

or, equivalently:

$$[[\mathbf{u}, \mathbf{v}], ...] = [[\mathbf{u}, ...], [\mathbf{v}, ...]]$$

The function "ad" acts in a manner that can be described with:

$$\begin{array}{ccccc} & & \mathbf{u} \wedge \mathbf{w} & & \\ & & \uparrow & & \\ \mathbf{u} \xrightarrow{ad} ad(\mathbf{u}) = & [\mathbf{u}, ...] = & \mathbf{u} \wedge ... & \longleftarrow & \mathbf{w} \end{array}$$

$$\begin{array}{ccccc} & & \mathbf{v} \wedge \mathbf{w} & & \\ & & \uparrow & & \\ \mathbf{v} \xrightarrow{ad} ad(\mathbf{v}) = & [\mathbf{v}, ...] = & \mathbf{v} \wedge ... & \longleftarrow & \mathbf{w} \end{array}$$

$$\begin{array}{ccccc} & & (\mathbf{u}\wedge\mathbf{v})\wedge\mathbf{w} & & \\ & & \uparrow & & \\ \mathbf{u}\wedge\mathbf{v} & \xrightarrow{ad} & ad(\mathbf{u}\wedge\mathbf{v}) = [\mathbf{u}\wedge\mathbf{v}, ...] & \longleftarrow & \mathbf{w} \end{array}$$

By the way, let also remark that:

$$\begin{array}{ccccc} & & \mathbf{u}\wedge(\mathbf{v}\wedge\mathbf{w}) & & \\ & & \uparrow & & \\ \mathbf{u} & \xrightarrow{ad} & ad(\mathbf{u}) = [\mathbf{u}, ...] & \longleftarrow & \mathbf{v}\wedge\mathbf{w} \end{array}$$

and:

$$\begin{array}{ccccc} & & \mathbf{v}\wedge(\mathbf{u}\wedge\mathbf{w}) & & \\ & & \uparrow & & \\ \mathbf{v} & \xrightarrow{ad} & ad(\mathbf{v}) = [\mathbf{v}, ...] & \longleftarrow & \mathbf{u}\wedge\mathbf{w} \end{array}$$

These diagrams call for comments. Neither the fourth nor the fifth can systematically be identified with the third one. The third one coincides with the fourth one only when the classical cross product is associative. In general, since the cross product is the prototype for any Lie bracket, the Jacobi's relation is true:

$$[\mathbf{u},\, [\mathbf{v},\, \mathbf{w}]] \;=\; [[\mathbf{u},\, \mathbf{v}],\, \mathbf{w}] \;+\; [\mathbf{v},\, [\mathbf{u},\, \mathbf{w}]]$$

and this fact gives the Leibniz relation again, explaining why $[\mathbf{u},\, ...]$ acts like a derivation by respect for $\mathbf{u}$; the remaining question concerning the existence of the morphism is:

$$\exists ?\, (\mathbf{u},\, \mathbf{v},\, \mathbf{w}) \,\in\, E^3(D,\, C):$$

$$[\mathbf{u},\, [\mathbf{v},\, \mathbf{w}]] \,-\, [\mathbf{v},\, [\mathbf{u},\, \mathbf{w}]] \;=\; [[\mathbf{u},\, \mathbf{w}],\, [\mathbf{v},\, \mathbf{w}]]$$

I give indications in the next subsection.

1.3 Extrapolation.

It is now easy to understand that the extrapolation of this discussion starts with:

$$ad([\mathbf{u},\, \mathbf{v}]_A) \;=\; ad_{[\mathbf{u},\, \mathbf{v}]_A} \;\equiv\; [ad_{\mathbf{u}},\, ad_{\mathbf{v}}]_A \;=\; [ad(\mathbf{u}),\, ad(\mathbf{v})]_A$$

$$ad_{[\mathbf{u},\, \mathbf{v}]_A} \;\equiv\; [ad_{\mathbf{u}},\, ad_{\mathbf{v}}]_A$$

Let consider $[[\mathbf{u},\, \mathbf{v}]_A,\, ...]_A$ and look if it can be related to $[\,[\mathbf{u},\, ...]_A,\, [\mathbf{v},\, ...]_A]_A$ when the Jacobi's relation is true; recall:

$$[\mathbf{u},\, [\mathbf{v},\, \mathbf{w}]_A]_A \;=\; [[\mathbf{u},\, \mathbf{v}]_A,\, \mathbf{w}]_A \,+\, [\mathbf{v},\, [\mathbf{u},\, \mathbf{w}]_A]_A$$

As a matter of facts, if the Jacobi's relation is true, then:

$$[\mathbf{u},\, [\mathbf{v},\, \mathbf{w}]_A]_A \,-\, [\mathbf{v},\, [\mathbf{u},\, \mathbf{w}]_A]_A \,=\, [[\mathbf{u},\, \mathbf{v}]_A,\, \mathbf{w}]_A$$

The underlying question is now:

$$\exists ?\, (\mathbf{u},\, \mathbf{v},\, \mathbf{w}) \,\in\, E^3(D,\, C):$$

$$[\mathbf{u},\, [\mathbf{v},\, \mathbf{w}]_A]_A \,-\, [\mathbf{v},\, [\mathbf{u},\, \mathbf{w}]_A]_A \,=\, [[\mathbf{u},\, \mathbf{w}]_A,\, [\mathbf{v},\, \mathbf{w}]_A]_A$$

Let remark:

1. That relation is trivially true for any $(\mathbf{u},\, \mathbf{u},\, \mathbf{w})$ because (i) $\mathbf{x}$ - $\mathbf{x} = \mathbf{0}$ for any $\mathbf{x}$ and (ii) $[...,\, ...]_A$ is a Lie bracket.

2. For any $(\mathbf{u},\, \mathbf{w},\, \mathbf{u})$ and any $(\mathbf{w},\, \mathbf{u},\, \mathbf{u})$, this relation writes respectively:

   $$[\mathbf{u},\, [\mathbf{w},\, \mathbf{u}]_A]_A \,-\, [\mathbf{w},\, [\mathbf{u},\, \mathbf{u}]_A]_A \,=\, [[\mathbf{u},\, \mathbf{u}]_A,\, [\mathbf{w},\, \mathbf{w}]_A]_A$$

   and:

   $$[\mathbf{w},\, [\mathbf{u},\, \mathbf{u}]_A]_A \,-\, [\mathbf{u},\, [\mathbf{w},\, \mathbf{u}]_A]_A \,=\, [[\mathbf{w},\, \mathbf{u}]_A,\, [\mathbf{u},\, \mathbf{u}]_A]_A$$

   Therefore, in both cases:

   $$[\mathbf{u},\, [\mathbf{u},\, \mathbf{w}]_A]_A \,=\, \mathbf{0}$$

   This is ressembling a vanishing second order derivation of $\mathbf{w}$ by respect for $\mathbf{u}$. For some authors, the vanishing of a second order derivation is the criterion justifying the label *intern derivation* for the first order derivation at hand; here $[\mathbf{u},\, ...]_A$. As a matter of facts, the element $(\mathbf{u},\, \mathbf{u},\, \mathbf{w})$ in E^3(D, C) validates the relation signing the morphism under study; its cyclic permutations -precisely $(\mathbf{w},\, \mathbf{u},\, \mathbf{u})$ and $(\mathbf{u},\, \mathbf{w},\, \mathbf{u})$- do not. But -simultaneously- they induce the vanishing double derivation by respect for $\mathbf{u}$. At this stage, it is hard to relate this result to a classical functional analysis.

1.4 The Leibniz rule

Let consider F(K^D; K), the set of all numerical functions acting simultaneously on D variables taken in K with a result (an image) in K. There is another general result in that branch of mathematics; as already remarked a long time ago, let suppose that the Leibniz is true for the elements in F(K^D; K), then:

$$\delta(f\,.\,g) \,=\, \delta(f)\,.\,g \,+\, f\,.\,\delta(g)$$

Definitions. (Generalized Leibniz rule. Relative derivation). If δ can be labeled, for example with another numerical function h, the Leibniz relation can be generalized:

$$\delta_h(f\,.\,g) \,=\, \delta_h(f)\,.\,g \,+\, f\,.\,\delta_h(g)$$

Intuitively, δ_h prefigures a derivation of ... by respect for h. It is a *relative derivation* when it acts ln respecting the relation:

$$\delta_h(f) \,=\, f\,.\,h \,-\, h\,.\,f$$

Remark. The relative derivation systematically vanishes when the numerical functions have their images in a set equipped with a commutative multiplication; for example in R or C. This is no more true when the images are in a set which is equipped with a non-commutative multiplication; e.g.: H or O.

Any way, with the proposed definition for a relative derivation:

- the Leibniz rule is always true in R and C because 0 = 0;
- in all set K equipped with a non-commutative multiplication, that definition allows the rewritting of the Leibniz relation as:

  $$(f\,.\,g)\,.\,h\,-\,h\,.\,(f\,.\,g)\,=\,(f\,.\,h\,-\,h\,.\,f)\,.\,g\,+\,f\,.\,(g\,.\,h\,-\,h\,.\,g)$$

  When, furthermore, the multiplication is (i) associative and (ii) distributive (on the left and on the right) on K:

  $$f\,.\,(g\,.\,h)\,-\,(h\,.\,f)\,.\,g\,=\,(f\,.\,h)\,.\,g\,-\,(h\,.\,f)\,.\,g\,+\,f\,.\,(g\,.\,h)\,-\,f\,.\,(h\,.\,g)$$
  $$\downarrow$$
  $$0\,=\,(f\,.\,h)\,.\,g\,-\,f\,.\,(h\,.\,g)$$

  It is easy to state that the Leibniz relation is a tautology (syn.: always true).

Theorem. Validity of the generalized Leibniz rule. The generalized Leibniz rule is true in any ring (K, .) when the relative derivation of an element, f, in that set by respect for any other one, h, is the braket [f, h] = δ_hf.

Corollary. The relative derivation by respect for the null element exists and always vanishes. This is due to the fact that forall (f. h): δ_hf = -δ_fh; hence for the pair (f, 0): δ_0f = -δ_f0 = 0.

1.5 The function Π^{-1}

Let now re-introduce the function Π^{-1} (Part 01, Subsection 2.3.1, pp. 22-23) with a tiny generalization translating the discussion initiated on C to any ring (K, .); recall: K is a abelian additive group equipped with an associative and a distributive multiplication.

Definition. The Π^{-1} function.

- The definition itself:

  $$\Pi^{-1}\,:\,F(K^D;\,K)\,\rightarrow\,V_D^-\,=\,\{E(D,\,K),\,\otimes_A,\,A\,\in\,C^-_{(D-D-D)}\}$$
  $$\exists!\,\mathbf{u}\,\in\,E(D,\,K)\,|\,f\,\rightarrow\,\Pi^{-1}(f)\,=\,\mathbf{u}$$

- Distributivity:

  $$\forall\,z\,\in\,K,\,f\,\in\,F(K^D;\,K):$$
  $$\Pi^{-1}(z\,.\,f)\,=\,z\,.\,\Pi^{-1}(f)$$

- Additive morphism:

  $$\forall\,f_1,\,f_2\,\in\,F(K^D;\,K):$$
  $$\Pi^{-1}(f_1\,+\,f_2)\,=\,\Pi^{-1}(f_1)\,+\,\Pi^{-1}(f_2)$$

- Multiplicative morphism:

  $$\forall\,f_1,\,f_2\,\in\,F(K^D;\,K):$$
  $$\Pi^{-1}(f_1\,.\,f_2)\,=\,\otimes_A(\Pi^{-1}(f_1),\,\Pi^{-1}(f_2))$$

- Behavior by respect for a derivation generically denoted with the symbol δ:

  $$\forall\,g\,\in\,F(K^D;\,K):$$
  $$\Pi^{-1}(\delta g)\,=\,\otimes_A(\mathbf{u},\,\Pi^{-1}(g))\,=\,[\mathbf{u},\,\Pi^{-1}(g)]_A$$

As already proved in part 01, the representation Π^{-1} is a morphism translating the Leibniz's rule acting on the elements in F(K^D; K) to the elements taken in V$^-{}_D$. This fact makes it easier to understand why [**u**, ...]$_A$ is a plausible representation for δ in V$^-{}_D$, hence a derivation.

1.6 Representations

Definition. Representation. Let recall some basics. Let $\hat{L}$ be some Lie-algebra, let V denote a K vector space and let σ be a linear application: $\hat{L} \to gl(\mathrm{V})$ such that:

$$\sigma([x, y]) = \sigma(x) \circ \sigma(y) - \sigma(y) \circ \sigma(x)$$

Then, per definition, (V, σ) is a representation of the Lie algebra $\hat{L}$. As a matter of facts, σ is a morphism between two Lie algebras.

Example. A associative tensor product deformed by an anti-symmetric cube. At this stage, let recall the first part of this memoir and the general discussion concerning the existence of a multiplicative morphism relating V_D to M(D, C); in peculiar, the one-to-one correspondance between the mathematical tool $\otimes_A(\mathbf{u}, ...)$ and the matrix ${}_A\Phi(\mathbf{u})$ in M(D, C). Let also recall that (i) the sets $\hat{V}_D$[2] and $\hat{M} = \{\mathrm{M(D, C)}, [..., ...]\}$ are two C-Lie algebras and that (ii) ${}_A\Phi$ is a linear function.

If the tensor product at hand, $\otimes_A$, is (i) deformed by an anti-symmetric cube A and (ii) associative, then ${}_A\Phi$ is a multiplicative morphism:

$${}_A\Phi(\otimes_A(\mathbf{u}, \mathbf{v})) = {}_A\Phi(\mathbf{u}) \,.\, {}_A\Phi(\mathbf{v})$$

But, in that case, in inverting the arguments:

$${}_A\Phi(\otimes_A(\mathbf{v}, \mathbf{u})) = {}_A\Phi(\mathbf{v}) \,.\, {}_A\Phi(\mathbf{u})$$

A substraction is giving:

$${}_A\Phi(\otimes_A(\mathbf{u}, \mathbf{v})) - {}_A\Phi(\otimes_A(\mathbf{v}, \mathbf{u})) = {}_A\Phi(\mathbf{u}) \,.\, {}_A\Phi(\mathbf{v}) - {}_A\Phi(\mathbf{v}) \,.\, {}_A\Phi(\mathbf{u})$$

Since ${}_A\Phi$ is a linear function:

$${}_A\Phi(\otimes_A(\mathbf{u}, \mathbf{v}) - \otimes_A(\mathbf{v}, \mathbf{u})) = [{}_A\Phi(\mathbf{u}), {}_A\Phi(\mathbf{v})]$$

Let recall that (i) here A is an element in $\mathrm{C}^-{}_{(D-D-D)}$ and (ii) the definition of any deformed Lie product; and get:

$${}_A\Phi([\mathbf{u}, \mathbf{v}]_A) = [{}_A\Phi(\mathbf{u}), {}_A\Phi(\mathbf{v})]$$

The pair $(\mathrm{V}^-{}_D, {}_A\Phi)$ is a representation of the Lie algebra $\hat{V}_D$. Furthermore, ${}_A\Phi$ is connecting two Lie algebras.

[2]The Lie algebra $\hat{V}_D$ is containing $\mathrm{V}^-{}_D$ but, at this stage, it is not known if both sets coincide; more work is needed.

Corollary. A fundamental property of Jacobi's territories with arguments in an associative $\mathbf{V}^{-}{}_D$. If the product ${}_A\otimes$ is associative and built with an anti-symmetric cube A, it generates a deformed Lie product (see Part 01, § Equivalence) and the Leibniz rule draws Jacobi's territories such that:

$$[\mathbf{u},\, [\mathbf{v},\, \mathbf{w}]_A]_A \;=\; [[\mathbf{u},\, \mathbf{v}]_A,\, \mathbf{w}]_A \,+\, [\mathbf{v},\, [\mathbf{u},\, \mathbf{w}]_A]_A$$

$$\downarrow$$

$$[[\mathbf{u},\, \mathbf{v}]_A,\, \mathbf{w}]_A \;=\; [[\mathbf{u},\, \mathbf{v}]_A,\, \mathbf{w}]_A \,+\, [\mathbf{v},\, [\mathbf{u},\, \mathbf{w}]_A]_A$$

$$\downarrow$$

$$[\mathbf{v},\, [\mathbf{u},\, \mathbf{w}]_A]_A \;=\; \mathbf{0}$$

And this relation is true for any cyclic and anti-cyclic permutation.

Example. The classical cross product. In that case, D = 3 and the cube A is reduced to a matrix [J] in M(3, F_3) where F_3 = {-1, 0, +1}. The previous relation writes:

$$\mathbf{v} \,\wedge\, (\mathbf{u} \,\wedge\, \mathbf{w}) \;=\; \mathbf{0}$$

This condition is realized when:

$$\exists\, \lambda \,\in\, K \,:\, \mathbf{v} \;=\; \lambda\,.\,(\mathbf{u} \,\wedge\, \mathbf{w})$$

But, since the discussion concerns an associative cross product, the previous relation can also be written:

$$(\mathbf{v} \,\wedge\, \mathbf{u}) \,\wedge\, \mathbf{w} \;=\; \mathbf{0}$$

This condition is realized when:

$$\exists\, \mu \,\in\, K \,:\, \mathbf{w} \;=\; \mu\,.\,(\mathbf{v} \,\wedge\, \mathbf{u})$$

Let inject the second condition into the first one:

$$\exists\,(\lambda,\, \mu) \,\in\, K^2 \,:\, \mathbf{v} \;=\; \lambda\,.\,\mu\,.\,(\mathbf{u} \,\wedge\, (\mathbf{v} \,\wedge\, \mathbf{u}))$$

Due to the classical rules, when $V^{-}{}_3$ is equipped with a Euclidean scalar product $<...,\, ...>_{Id3}$, the resulting condition is:

$$\exists\,(\lambda,\, \mu) \,\in\, K^2 \,:\, (1 \,-\, \lambda\,.\,\mu\,.\, <\mathbf{u},\, \mathbf{u}>_{Id_3})\,.\,\mathbf{v} \;=<\mathbf{u},\, \mathbf{v}>_{Id_3}\,.\,\mathbf{u}$$

The arguments **u** and **v** are colinears. Hence, the argument **w** vanishes because of the second condition. Because of the first condition, the argument **v** vanishes too. At the end of this exploration, (**0**, **0**, **0**) is the unique element in the Jacobi's territory related to a classical and associative $V^{-}{}_3$.

Definition. The function Ψ. Let extrapolate the discussion concerning the set of numerical functions to the one of matrices in M(D, K). Let recall that the set {M(D, K), .} is a non-comutative ring. Let imagine the existence of $\hat{\Psi}$ = F(F(K^D; K); M(D, K)), the set of all functions connecting the numerical functions to (D-D) matrices with components in K.

$$\hat{\Psi} \;=\; F(F(K^D;K);M(D,K)) \,:$$

$$\{\Psi \,:\, f \,\in\, F(K^D;K) \,\rightarrow\, \Psi(f) \;=\; [M_f] \,\in\, M(D,\, K)\}$$

Let suppose that the Ψs are linear functions defining a double morphism (additive and multiplicative) such that:

$$\forall\, f_1,\, f_2 \,\in\, F(K^D;K) \,:$$

$$\Psi(f_1 + f_2) = [M_{f_1 + f_2}] = [M_{f_1}] + [M_{f_2}] = \Psi(f_1) + \Psi(f_2)$$
$$\Psi(f_1 \,.\, f_2) = [M_{(f_1 \,.\, f_2)}] = [M_{f_1}] \,.\, [M_{f_2}] = \Psi(f_1) \,.\, \Psi(f_2)$$

Remark. Representing the relative derivation. Let now come back to the previous Subsection 1.1.4. Let recall the definition of the relative derivation:

$$\delta_h(f) = f \,.\, h - h \,.\, f$$

Let project it in M(D, K) with Ψ and easily get the relation:

$$\Psi(\delta_h(f)) = [M_f] \,.\, [M_h] - [M_h] \,.\, [M_f] = [[M_f], [M_h]]$$

Proposition: Let suppose that:

$$\Psi = {}_A\Phi \circ \Pi^{-1}$$

And let represent any f:

$$\forall f \in F(K^D; K), \exists! \, \mathbf{u} \in E(D, K) :$$
$$\Psi(f) = ({}_A\Phi \circ \Pi^{-1})(f) = {}_A\Phi((\Pi^{-1})(f))) = {}_A\Phi(\mathbf{u})$$

As consequence, get for any other h:

$$h \in F(K^D; K), \exists! \, \mathbf{v} \in E(D, K) :$$
$$\Psi(h) = ({}_A\Phi \circ \Pi^{-1})(h) = {}_A\Phi((\Pi^{-1})(h))) = {}_A\Phi(\mathbf{v})$$

Hence, when the proposition connecting Ψ and Π^{-1} is true, there exists at least one trivial decomposition representing the generic matrix [M_f]; in that context, with that proposition:

$$\exists! \, (\mathbf{u}, \mathbf{v}) :$$
$$\Psi(\delta_h(f)) = [[M_f], [M_h]] = [{}_A\Phi(\mathbf{u}), {}_A\Phi(\mathbf{v})]_A$$

Let now consider the matrix:

$${}_A\Phi([\mathbf{u}, \mathbf{v}]_A)$$

As soon as the function Π^{-1} *exists and is surjective*, it can be induced that each pair of vectors results from the existence of a pair of functions:

$${}_A\Phi([\mathbf{u}, \mathbf{v}]_A) = {}_A\Phi([\Pi^{-1}(f_1), \Pi^{-1}(f_2)]_A)$$

Because of the definition of the deformed Lie product:

$${}_A\Phi([\mathbf{u}, \mathbf{v}]_A) = {}_A\Phi(\otimes_A(\Pi^{-1}(f_1), \Pi^{-1}(f_2) - \otimes_A(\Pi^{-1}(f_2), \Pi^{-1}(f_1))$$

Because ${}_A\Phi$ is linear:

$${}_A\Phi([\mathbf{u}, \mathbf{v}]_A) = {}_A\Phi(\otimes_A(\Pi^{-1}(f_1), \Pi^{-1}(f_2)) - {}_A\Phi(\otimes_A(\Pi^{-1}(f_2), \Pi^{-1}(f_1))$$

Because Π^{-1} is a multiplicative morphism (see Subsection1.1.5):

$${}_A\Phi([\mathbf{u}, \mathbf{v}]_A) = {}_A\Phi(\Pi^{-1}(f_1 \,.\, f_2)) - {}_A\Phi(\Pi^{-1}(f_2 \,.\, f_1))$$

Again because ${}_A\Phi$ is linear:

$${}_A\Phi([\mathbf{u}, \mathbf{v}]_A) = {}_A\Phi(\Pi^{-1}(f_1 \,.\, f_2) - \Pi^{-1}(f_2 \,.\, f_1))$$

Because Π^{-1} is linear:

$${}_A\Phi([\mathbf{u}, \mathbf{v}]_A) = {}_A\Phi(\Pi^{-1}(f_1 \,.\, f_2 - f_2 \,.\, f_1))$$

Because of the relation defining a relative derivation:

$$_A\Phi([\mathbf{u},\,\mathbf{v}]_A) \;=\; _A\Phi(\Pi^{-1}(\delta_{f_2}(f_1)))$$

If the relation connecting Ψ to $_A\Phi$ and Π^{-1} is true (the above proposition), then:

$$_A\Phi([\mathbf{u},\,\mathbf{v}]_A) \;=\; \Psi(\delta_{f_2}(f_1))$$

Since, in that case, the r.h.t is also a representation for a Lie bracket connecting two trivial matrices (see the starting point of this specific investigation):

$$_A\Phi([\mathbf{u},\,\mathbf{v}]_A) \;=\; [_A\Phi(\mathbf{u}),\, _A\Phi(\mathbf{v})]_A$$

and $_A\Phi$ is a morphism.

Theorem. Representing the relative derivations. When Π^{-1} *exists and is surjective*, if the relation connecting Ψ to $_A\Phi$ and Π^{-1} is true:

$$\Psi \;=\; _A\Phi \circ \Pi^{-1}$$

then $_A\Phi$ is a morphism. The set {F(K^D; K), [..., ...]} is a Lie algebra as long as (K, .) is a ring; $\hat{M}$ is a Lie algebra too. The pair (F(K^D; K), Ψ) is a representation of {F(K^D; K), [..., ...]} in $\hat{M}$.

1.7 Discussion about Π^{-1}.

What does the function Π^{-1} represent? If its inverse, Π, exists, then it represents a function relating a vector in E(D, K) (for example: **u**) to a numerical function in F(K^D; K) (for example: f). But, what does that kind of relation really mean? In which circumstances do we relate a vector and a function? Why? With which purpose?

Example - Polynomials with at least one solution. There is a certitude: that type of function exists. For example let consider a polynomial f in F(K^D; K) and let look for its solutions. What does happen in such circumstances? Answer: we develop methods allowing the discovery of elements in K^D for which f vanishes. Since K^D is isomorphic to E(D, K), the quest is equivalent to the discovery of at least one element **u** in E(D, K) such that f(**u**) = 0. Within that context, there is a set of functions Π^{-1} relating polynomials to their solutions. These functions exist as long as we limit the discussion to a set of polynomials that have at least one solution.

Example - The theory of the (E) Question. There is at least another example illustrating the diverse roles that can be accomplished by Π^{-1}. The initial formulation of the so-called (E) question is contained in the following sentence:

$$A \;\in\; C^{-}_{D-D-D}, (\mathbf{u},\,\mathbf{v}) \in E^2(D,K):$$

$$\exists!?\,([P],\,\mathbf{z}) \in M(D,K) \times E(D,K), |[\mathbf{u},\,\mathbf{v}]_A >= [P]\,.\,|\mathbf{v} > +|\mathbf{z} >$$

Former investigations have proved that the question accepts at least one positive answer; this is the trivial decomposition ($_A\Phi(\mathbf{u})$, **0**). The intellectual curiosity pushes to ask if there is another solution? To investigate this question, let suppose that E(D, K) is equipped with a Euclidean scalar product. With that hypothesis, three elements in K can be built; they are precisely:

Definition. The scalar associated with the projectile.

$$\exists\,([P],\,\mathbf{z}) \in M(D,K) \times E(D,K), |[\mathbf{u},\,\mathbf{v}]_A >= [P]\,.\,|\mathbf{v} > +|\mathbf{z} >$$

$$\downarrow$$

$$\exists\, p(\mathbf{u}) \;=< \mathbf{u},\, |[\mathbf{u},\, \mathbf{v}]_A > \;- \{[P]\,.\,|\mathbf{v} > \;+ |\mathbf{z} >\} >_{Id_D} \in\, K$$

Definition. The scalar associated with the target.

$$\exists\, ([P],\, \mathbf{z}) \in M(D,K) \times E(D,K), |[\mathbf{u},\, \mathbf{v}]_A >= \,[P]\,.\,|\mathbf{v} > \,+ |\mathbf{z} >$$

$$\downarrow$$

$$\exists\, t(\mathbf{v}) \;=< \mathbf{v},\, |[\mathbf{u},\, \mathbf{v}]_A > \;- \{[P]\,.\,|\mathbf{v} > \;+ |\mathbf{z} >\} >_{Id_D} \in\, K$$

Definition. The scalar associated with the residual part.

$$\exists\, ([P],\, \mathbf{z}) \in M(D,K) \times E(D,K), |[\mathbf{u},\, \mathbf{v}]_A >= \,[P]\,.\,|\mathbf{v} > \,+ |\mathbf{z} >$$

$$\downarrow$$

$$\exists\, r(\mathbf{z}) \;=< \mathbf{z},\, |[\mathbf{u},\, \mathbf{v}]_A > \;- \{[P]\,.\,|\mathbf{v} > \;+ |\mathbf{z} >\} >_{Id_D} \in\, K$$

The important point, here, is the fact that a positive answer to the (E) question is accompanied by the existence of three vanishing functions, p, t and r, respectively related to the vectors **u**, **v** and **z** (projectile, target and residual part).

Each of these functions can be understood as a particular visage for Π; the existence of a corresponding visage for Π^{-1} is clearly related to the inversibility of these functions.

Since there always exists at least one trivial decomposition for each deformed Lie product, the inversibility depends on our ability to isolate at least one vector (synonym: to find at least one solution) for each of the three polynomials:

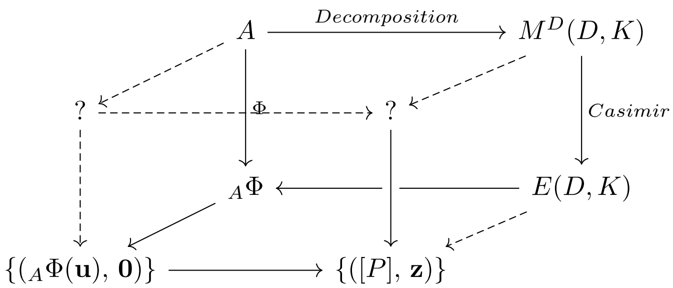
- $$\exists\, \mathbf{u}?\,|\,\Pi(\mathbf{u}) \;=< \mathbf{u},\, |\{_A\Phi(\mathbf{u}) \,-\, [P]\}\,.\,|\mathbf{v} > \,- |\mathbf{z} >>_{Id_D} = \,0$$
  when **v**, ([P], **z**) are known.
- $$\exists\, \mathbf{v}?\,|\,\Pi(\mathbf{v}) \;=< \mathbf{v},\, |\{_A\Phi(\mathbf{u}) \,-\, [P]\}\,.\,|\mathbf{v} > \,- |\mathbf{z} >>_{Id_D} = \,0$$
  when **u**, ([P], **z**) are known.
- $$\exists\, \mathbf{z}?\,|\,\Pi(\mathbf{z}) \;=< \mathbf{z},\, |\{_A\Phi(\mathbf{u}) \,-\, [P]\}\,.\,|\mathbf{v} > \,- |\mathbf{z} >>_{Id_D} = \,0$$
  when **u**, **v** and [P] are known.

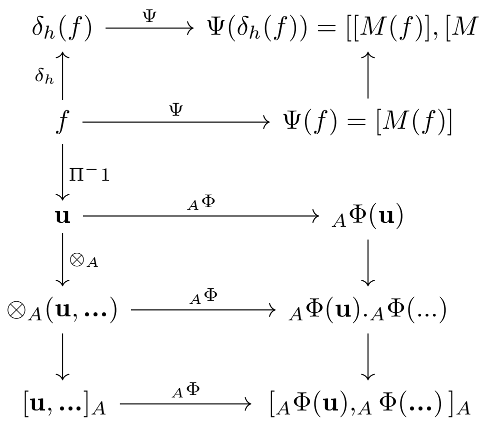
In all cases, the goal is the discovery of solutions for a quadratic form. Each time there exists a solution, Π^{-1} exists too; its function can be described as: "Give me at least the formalism of one of the three quadratic forms (p, t or r) that can be associated with a generic decomposition of a given deformed Lie product and I tell you its solutions." Concretely, the function Π^{-1} acts a little bit like a computer program serving the roots of a given polynomial.

Having said that, the obligatory existence of at least one trivial decomposition irremediably connects the corresponding Π^{-1} function to the inversibility of $_A\Phi$ or, equivalently, to its surjectivity. The question has been answered in the first part of this memoir.

1.8 Structuring the discussion.

I can now propose two provisory schemes sketching the relations between the mathematical objects which have been introduced until here into that dissertation (Part 01 and 02):





1.9 Facts and questions.

The progression has brought the following facts:

- The deformed Lie products can be represented in $\hat{M}$.
- The relative derivations as well.
- The extrinsic and the intrinsic method suggest the existence of non-trivial decompositions.
- If Π^{-1} is a surjection and if $\Psi(\mathrm{f}) = {}_A\Phi(\mathbf{u})$, then ${}_A\Phi$ is a morphism.

They induce the next questions:

- **Question 01.** Is there a link between the set of all relative derivations and the one of deformed Lie products? If yes, which one?
- **Question 02.** It may now be asked if the projection of a variation δ of a given numerical function denoted ... via Π^{-1} is equivalent to varying the projection Π^{-1} acting on that function via a generic $\hat{\delta}$ function for which a representation is yet needed?

$$\Pi^{-1}(\delta...) = [\mathbf{u}, ...]_A = (\hat{\delta}\Pi^{-1})(...)$$

- **Question 03.** How can we translate the construction of non-trivial decompositions into a logical schema?

$$
\begin{array}{ccc}
\delta_h(p) & \xrightarrow{\Psi} & \Psi(\delta_h(p)) = [[M(p)], [M(h)]] \\
\uparrow{\scriptstyle \delta_h} & & \uparrow{\scriptstyle ?} \\
p & \xrightarrow{\Psi} & \Psi(p) = [P] =_A \Phi(\mathbf{u}) - [B]^{-1}.Hess_{...}t(...) \\
\uparrow{\scriptstyle \Pi} & & \\
\mathbf{u} & \xrightarrow{{}_A\Phi} & {}_A\Phi(\mathbf{u}) \\
\downarrow{\scriptstyle \otimes_A} & & \downarrow \\
\otimes_A(\mathbf{u}, \mathbf{...}) & \xrightarrow{{}_A\Phi} & {}_A\Phi(\mathbf{u}).{}_A\Phi(...) \\
\downarrow & & \downarrow \\
[\mathbf{u}, \mathbf{...}]_A & \xrightarrow{{}_A\Phi} & [{}_A\Phi(\mathbf{u}), {}_A\Phi(\mathbf{...})]_A
\end{array}
$$

Provisory end.