New analysis of the Klein-Gordon equation within the theory of deformed cross products. Part III: Tools for a quantization.

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This is the third part in a series of explorations analyzing the Klein-Gordon equation. The document applies a generic procedure of quantization to Kerns appearing when deformed cross products, especially angular momentum, are decomposed with the help of the mathematical method which has been exposed in [[a]]. The first calculations are encouraging; they suggest interesting tools that I shall try to involve later in the construction of a theory of quantum gravity.

Key words : Quantization, tools, deformed cross products, angular momentum.

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1 Axiomatic for a quantification of the theory.

1.1 Introducing the first stones of the procedure.

Remark 1.1. *Motivations.*

Because:

1. Angular momenta are quantified (this is really not a scoop);
2. Former parts of this global analysis concerning the Klein-Gordon equation has convinced me that a non-trivially decomposed angular momentum may eventually be related to the concept of spin that has been introduced by Dirac, Weyl and many others in quantum theory; see [Part II: Identifications, sub-section 1.6, pp. 10-12].

It is meaningful to introduce fundamental and basic concepts of quantum mechanics into this new approach.

Remark 1.2. *Reformulating the classical angular momentum.*

Let:

- recall the intrinsic method [[a]] and, accompanying it, the fact that any decomposition is related to a polynomial Λ;
- consider a generic angular momentum $\mathbf{J}^*$ such that[1]:

$$|\mathbf{J}^* > = \hbar \,.\, |[\mathbf{k}^*, \mathbf{x}^*]_{[A]} > = \hbar \,.\, [P] \,.\, |\mathbf{x}^* > + \hbar \,.\, |\mathbf{z}^* >$$

- look for its possible non-trivial decomposition. The main part of this decomposition, [P], has a generic formalism depending on the degeneracy (type II) or on the non-degeneracy (type I) of the polynomial Λ; precisely:
  - For the type I [[a]; p. 25]:

$$[P_{I,\,|A|}] \,=\, \{[A]^t \,.\, [J]\} \,.\, \{\frac{|A|}{2} \,.\, [Hess_{\mathbf{k}}\Lambda] \,-\, {}_{[J]}\Phi({}^{(3)}\mathbf{s}_\Lambda)\},\; |A| \,=\, \pm 1$$

  - For the type II, there exists a pair $(\mathbf{a},\, \mathbf{b})$ in E(3, C) $\times$ E(3, c) such that:

$$[P_{II}] = \{[A]^t \,.\, [J]\} \,.\, \{\frac{1}{2} \,.\, \{T_2(\otimes)(\mathbf{a},\, \mathbf{b}) \,+\, T_2^t(\otimes)(\mathbf{a},\, \mathbf{b})\} \,+\, {}_{[A]}\Phi(\frac{1}{2} \,.\, ({}^{(3)}\mathbf{a} \wedge \mathbf{b}))\}$$

    and where the first sum plays the role of the Hessian.

  A classical result of that toy theory is contained in that equation:

$$|[\mathbf{q}_1, \mathbf{q}_2]_{[A]} > = [A]^t \,.\, [J] \,.\, |\mathbf{q}_1 \wedge \mathbf{q}_2 > \,;\; |A| \,=\, \pm 1$$

  with:

$$[J] \,=\, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \,;\; [J]^t \,=\, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

[1] Recall: in my work, the asterisk denotes the co-variant version of the mathematical object preceding it.

Since the deforming matrix [A] and its Euclidean limit [J] are non-degenerated matrices, it is also always possible to write:

$$\{[A]^t \,.\, [J]\}^{-1} \,.\, |[\mathbf{q}_1, \mathbf{q}_2]_{[A]} >= |\mathbf{q}_1 \wedge \mathbf{q}_2 >$$

In the case at hand, the very classical angular momentum can be decomposed as:

$$|\mathbf{J}_0 >= \hbar \,.\, \{[A]^t \,.\, [J]\}^{-1} \,.\, \{[P] \,.\, |\mathbf{x}^* > + |\mathbf{z}^* >\}$$

For a non-degenerated polynomial:

$$|\mathbf{J}_0 >$$

$$=$$

$$\hbar \,.\, \{[A]^t \,.\, [J]\}^{-1} \,.\, \{\{[A]^t \,.\, [J]\} \,.\, \{\frac{|A|}{2} \,.\, [Hess_{\mathbf{k}}\Lambda] - {}_{[J]}\Phi({}^{(3)}\mathbf{s}_\Lambda)\} \,.\, |\mathbf{x}^* > + |\mathbf{z}^* >\}$$

$$=$$

$$\hbar \,.\, \{\{\frac{|A|}{2} \,.\, [Hess_{\mathbf{k}}\Lambda] - {}_{[J]}\Phi({}^{(3)}\mathbf{s}_\Lambda)\} \,.\, |\mathbf{x}^* > + \{[A]^t \,.\, [J]\}^{-1} \,.\, \mathbf{z}^* >\}$$

$$=$$

$$\hbar \,.\, \{K([P]_{|A|}) \,.\, |\mathbf{x}^* > + |\mathbf{Z}^* >\}$$

Please note that any classical angular momentum $\mathbf{J}_0$ can be decomposed, eventually non-trivially; the main part of that decomposition is the Kern:

$$K([P]_{|A|}) = \{\frac{|A|}{2} \,.\, [Hess_{\mathbf{k}}\Lambda] - {}_{[J]}\Phi({}^{(3)}\mathbf{s}_\Lambda)\}, \ |A| = \pm 1$$

Its residual part is:

$$|\mathbf{Z}^* >= \{[A]^t \,.\, [J]\}^{-1} \,.\, \mathbf{z}^* >$$

Remark 1.3. *The aim of that document.*

In a first step, following the axiomatic which is exposed in [[13]; §1.2.2], I examine to what extent the set of the main parts of the non-trivial decomposition can form a Hilbert space.

- Axiom A1: This axiom imposes to work with vectors. It is well-accepted and known that M(3, C) is a vector space; each main part is an element in M(3, C), hence it is a vector.

- Axiom A2: at this stage, I presume that the angular momentum $\mathbf{J}$ is the observable physical quantities which are represented by the main parts.

- Axiom A3: Diverse brackets involving pairs of Kern can easily be formed and then be calculated. I develop that point in details in the next section.

- Axiom A4 and A5 will be introduced and commented later.

The next section explores that vision in more details.

1.2 Calculating diverse brackets.

Remark 1.4. *The anti-commutator $\{K, K^t\}$.*

Let focus the attention on the non-degenerated polynomials Λ and denote their subset with Pol(2)E(3, C)*. Then, inspired by general considerations which are for example also exposed in [[12]; especially the relations p. 7, (2a.6)], I write in peculiar:

$$\forall\,\Lambda \in Pol(2)E(3, C)^* : \uparrow(\Lambda) = K_\Lambda([P]_{|A|}) = |A| \,.\, \{\frac{1}{2} \,.\, [H_\Lambda] - |A| \,.\, {}_{[J]}\Phi({}_\Lambda\mathbf{s})\} \quad (1)$$

$$\forall\,\Lambda \in Pol(2)E(3, C)^* : \downarrow(\Lambda) = K^t_\Lambda([P]_{|A|}) = |A| \,.\, \{\frac{1}{2} \,.\, [H_\Lambda] + |A| \,.\, {}_{[J]}\Phi({}_\Lambda\mathbf{s})\} \quad (2)$$

Let calculate this first generic bracket:

$$\forall\,\Lambda_1,\, \Lambda_2 \in Pol(2)E(3, C)^*, \forall \downarrow(\Lambda_1), \uparrow(\Lambda_2) \in M(3, C) :$$

$$\{\downarrow(\Lambda_1), \uparrow(\Lambda_2)\}$$
$$=$$
$$\downarrow(\Lambda_1) \,.\, \uparrow(\Lambda_2) + \uparrow(\Lambda_2) \,.\, \downarrow(\Lambda_1)$$
$$=$$
$$\{\frac{1}{2} \,.\, [H_{\Lambda_1}] + |A| \,.\, {}_{[J]}\Phi({}_{\Lambda_1}\mathbf{s})\} \,.\, \{\frac{1}{2} \,.\, [H_{\Lambda_2}] - |A| \,.\, {}_{[J]}\Phi({}_{\Lambda_2}\mathbf{s})\}$$
$$+\{\frac{1}{2} \,.\, [H_{\Lambda_2}] - |A| \,.\, {}_{[J]}\Phi({}_{\Lambda_2}\mathbf{s})\} \,.\, \{\frac{1}{2} \,.\, [H_{\Lambda_1}] + |A| \,.\, {}_{[J]}\Phi({}_{\Lambda_1}\mathbf{s})\}$$
$$=$$
$$\frac{1}{4} \,.\, \{[H_{\Lambda_1}] \,.\, [H_{\Lambda_2}] + [H_{\Lambda_2}] \,.\, [H_{\Lambda_1}]\} - \{{}_{[J]}\Phi({}_{\Lambda_1}\mathbf{s}) \,.\, {}_{[J]}\Phi({}_{\Lambda_2}\mathbf{s}) + {}_{[J]}\Phi({}_{\Lambda_2}\mathbf{s}) \,.\, {}_{[J]}\Phi({}_{\Lambda_1}\mathbf{s})\}$$
$$+\frac{|A|}{2} \,.\, \{-[H_{\Lambda_1}] \,.\, {}_{[J]}\Phi({}_{\Lambda_2}\mathbf{s}) + {}_{[J]}\Phi({}_{\Lambda_1}\mathbf{s}) \,.\, [H_{\Lambda_2}] + [H_{\Lambda_2}] \,.\, {}_{[J]}\Phi({}_{\Lambda_1}\mathbf{s}) - {}_{[J]}\Phi({}_{\Lambda_2}\mathbf{s}) \,.\, [H_{\Lambda_1}]\}$$

In peculiar, when both polynomials coincide:

$$\{\downarrow(\Lambda), \uparrow(\Lambda)\} = \downarrow(\Lambda) \,.\, \uparrow(\Lambda) + \uparrow(\Lambda) \,.\, \downarrow(\Lambda) = \frac{1}{2} \,.\, \{[H_\Lambda]^2 - {}_{[J]}\Phi^2(\mathbf{s}_\Lambda)\} \qquad (3)$$

Hence, this bracket vanishes when:

$$[H_\Lambda]^2 = {}_{[J]}\Phi^2(\mathbf{s}_\Lambda)$$

Remark 1.5. *The commutator $[K, K^t]$.*

After this warm-up, let follow the vein in calculating that other generic bracket:

$$\forall\,\Lambda_1,\, \Lambda_2 \in Pol(2)E(3, C)^*, \forall \downarrow(\Lambda_1), \uparrow(\Lambda_2) \in M(3, C) :$$

$$[\downarrow(\Lambda_1), \uparrow(\Lambda_2)] \qquad (4)$$
$$=$$
$$\downarrow(\Lambda_1) \,.\, \uparrow(\Lambda_2) - \uparrow(\Lambda_2) \,.\, \downarrow(\Lambda_1)$$
$$=$$
$$\{\frac{1}{2} \,.\, [H_{\Lambda_1}] + |A| \,.\, {}_{[J]}\Phi({}_{\Lambda_1}\mathbf{s})\} \,.\, \{\frac{1}{2} \,.\, [H_{\Lambda_2}] - |A| \,.\, {}_{[J]}\Phi({}_{\Lambda_2}\mathbf{s})\}$$
$$-\{\frac{1}{2} \,.\, [H_{\Lambda_2}] - |A| \,.\, {}_{[J]}\Phi({}_{\Lambda_2}\mathbf{s})\} \,.\, \{\frac{1}{2} \,.\, [H_{\Lambda_1}] + |A| \,.\, {}_{[J]}\Phi({}_{\Lambda_1}\mathbf{s})\}$$

$$=$$

$$\frac{1}{4} \cdot \{[H_{\Lambda_1}] \, . \, [H_{\Lambda_2}] \; - \; [H_{\Lambda_2}] \, . \, [H_{\Lambda_1}]\} \; + \; \{{}_{[J]}\Phi({}_{\Lambda_1}\mathbf{s}) \, . \, {}_{[J]}\Phi({}_{\Lambda_2}\mathbf{s}) \; + \; {}_{[J]}\Phi({}_{\Lambda_2}\mathbf{s}) \, . \, {}_{[J]}\Phi({}_{\Lambda_1}\mathbf{s})\}$$

$$+ \frac{|A|}{2} \cdot \{-[H_{\Lambda_1}] \, . \, {}_{[J]}\Phi({}_{\Lambda_2}\mathbf{s}) + {}_{[J]}\Phi({}_{\Lambda_1}\mathbf{s}) \, . \, [H_{\Lambda_2}] - [H_{\Lambda_2}] \, . \, {}_{[J]}\Phi({}_{\Lambda_1}\mathbf{s}) + {}_{[J]}\Phi({}_{\Lambda_2}\mathbf{s}) \, . \, [H_{\Lambda_1}]\}$$

In peculiar, when both polynomials coincide:

$$[\downarrow(\Lambda), \uparrow(\Lambda)] \tag{5}$$

$$=$$

$$\downarrow(\Lambda) \, . \, \uparrow(\Lambda) - \uparrow(\Lambda) \, . \, \downarrow(\Lambda)$$

$$=$$

$$\frac{1}{2} \cdot {}_{[J]}\Phi^2(\mathbf{s}_\Lambda) \; + \; |A| \, . \, \{{}_{[J]}\Phi({}_\Lambda\mathbf{s}) \, . \, [H_\Lambda] \; - \; [H_\Lambda] \, . \, {}_{[J]}\Phi({}_\Lambda\mathbf{s})\}$$

$$=$$

$$\frac{1}{2} \cdot {}_{[J]}\Phi^2(\mathbf{s}_\Lambda) \; + \; |A| \, . \, [{}_{[J]}\Phi({}_\Lambda\mathbf{s}), \, [H_\Lambda]]$$

Hence, this bracket vanishes when:

$$\frac{1}{2} \cdot {}_{[J]}\Phi^2(\mathbf{s}_\Lambda) \; + \; |A| \, . \, [{}_{[J]}\Phi({}_\Lambda\mathbf{s}), \, [H_\Lambda]] \; = \; {}^{(3)}[0] \tag{6}$$

Remark 1.6. *The anti-commutator* $\{\Phi, \, H\}$.

Let calculate for further explorations:

$$[H_\Lambda] \, . \, {}_{[J]}\Phi({}_\Lambda\mathbf{s}) \tag{7}$$

$$=$$

$$\begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} . \begin{bmatrix} 0 & -s^3 & s^2 \\ s^3 & 0 & -s^1 \\ -s^2 & s^1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} H_{12} \, . \, s^3 \; - \; H_{13} \, . \, s^2 & H_{13} \, . \, s^1 \; - \; H_{11} \, . \, s^3 & H_{11} \, . \, s^2 \; - \; H_{12} \, . \, s^1 \\ H_{22} \, . \, s^3 \; - \; H_{23} \, . \, s^2 & H_{23} \, . \, s^1 \; - \; H_{21} \, . \, s^3 & H_{21} \, . \, s^2 \; - \; H_{22} \, . \, s^1 \\ H_{32} \, . \, s^3 \; - \; H_{33} \, . \, s^2 & H_{33} \, . \, s^1 \; - \; H_{31} \, . \, s^3 & H_{31} \, . \, s^2 \; - \; H_{32} \, . \, s^1 \end{bmatrix}$$

On the same vein:

$${}_{[J]}\Phi({}_\Lambda\mathbf{s}) \, . \, [H_\Lambda] \tag{8}$$

$$=$$

$$\begin{bmatrix} 0 & -s^3 & s^2 \\ s^3 & 0 & -s^1 \\ -s^2 & s^1 & 0 \end{bmatrix} . \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} H_{31} \, . \, s^2 \; - \; H_{21} \, . \, s^3 & H_{32} \, . \, s^2 \; - \; H_{22} \, . \, s^3 & H_{33} \, . \, s^2 \; - \; H_{23} \, . \, s^3 \\ H_{11} \, . \, s^3 \; - \; H_{31} \, . \, s^1 & H_{12} \, . \, s^3 \; - \; H_{32} \, . \, s^1 & H_{13} \, . \, s^3 \; - \; H_{33} \, . \, s^1 \\ H_{21} \, . \, s^1 \; - \; H_{11} \, . \, s^2 & H_{22} \, . \, s^1 \; - \; H_{12} \, . \, s^2 & H_{23} \, . \, s^1 \; - \; H_{13} \, . \, s^2 \end{bmatrix}$$

Since any Hessian of a smooth function is supposed to be a symmetric matrix:

$$\forall \, a, \, b \; = \; 1, 2, 3 : \; H_{ab} \; = \; H_{ba} \tag{9}$$

This yields:

$$\{[H_\Lambda] \, . \, {}_{[J]}\Phi({}_\Lambda\mathbf{s}) \; + \; {}_{[J]}\Phi({}_\Lambda\mathbf{s}) \, . \, [H_\Lambda]\} \tag{10}$$

$$=$$

$$\begin{bmatrix} 0 & \sum_a H_{a3} \,.\, s^a \,-\, Tr[H_\Lambda] \,.\, s^3 & Tr[H_\Lambda] \,.\, s^2 \,-\, \sum_a H_{a2} \,.\, s^a \\ Tr[H_\Lambda] \,.\, s^3 \,-\, \sum_a H_{a3} \,.\, s^a & 0 & \sum_a H_{a1} \,.\, s^a \,-\, Tr[H_\Lambda] \,.\, s^1 \\ \sum_a H_{a2} \,.\, s^a \,-\, Tr[H_\Lambda] \,.\, s^2 & Tr[H_\Lambda] \,.\, s^1 \,-\, \sum_a H_{a1} \,.\, s^a & 0 \end{bmatrix}$$

This matrix is anti-symmetric; let introduce the components of an unknown vector **X**:

$$\forall\, a,\, b \,=\, 1,2,3 :\, {}_\Lambda X^b \,=\, Tr[H_\Lambda] \,.\, s^b \,-\, \sum_a H_{ab} \,.\, s^a$$

More concisely, this is also nothing but:

$$|_\Lambda \mathbf{X} >= \{Tr[H_\Lambda] \,.\, Id_3 \,-\, [H_\Lambda]\} \,.\, |_\Lambda \mathbf{s} > \qquad (11)$$

Hence, the (symmetric) anti-commutator {Φ, H} is:

$$\{{}_{[J]}\Phi({}_\Lambda\mathbf{s}),\, [H_\Lambda]\} \,=\, \{[H_\Lambda] \,.\, {}_{[J]}\Phi({}_\Lambda\mathbf{s}) \,+\, {}_{[J]}\Phi({}_\Lambda\mathbf{s}) \,.\, [H_\Lambda]\} \,=\, {}_{[J]}\Phi({}_\Lambda\mathbf{X}) \qquad (12)$$

I think that it is worthwhile to remark the similitude between the formalism of this anti-commutator and the quite more general one of the Hamiltonian formalism. I mean that if, especially here, the Hessian would be the representation for some adequate Hamiltonian, then it would be meaningful to interpret that bracket as a mathematical tool revealing the evolution of the rotation matrix by respect for the time; precisely (see for example [[13]; exercise 1.1, (1.23)]):

$$\{{}_{[J]}\Phi({}_\Lambda\mathbf{s}),\, [H_\Lambda]\} \,=\, \frac{d_{[J]}\Phi({}_\Lambda\mathbf{s})}{dt} \,=\, {}_{[J]}\Phi({}_\Lambda\mathbf{X}) \qquad (13)$$

With that interpretation, observing attentively the relation defining the vector **X** - the Equ.(11), it becomes obvious that the matrix acting on the left side of the singular vector ${}_\Lambda\mathbf{s}$ plays a role equivalent to a variation by respect for the time:

$$\frac{d...}{dt} \,\equiv\, \{Tr[H_\Lambda] \,.\, Id_3 \,-\, [H_\Lambda]\} \qquad (14)$$

That matrix can be interpreted as a representation of the consequence of: "becoming older", or of "the flow of time". In fact, considering the third axiom of a canonical quantification, the Equ.(13) can be seen as a peculiar representation of a Heisenberg's equation of motion [[13]; A3, (1.34)]. Furthermore - and quite interesting for future physical applications- the eigenvalues of that matrix (up to a minus sign) are the values of the classical Laplace operator ΔΛ because:

Remark 1.7. *The trace of a classical Hessian matrix.*

The trace of any classical Hessian matrix coincides with the Laplace operator of the function of which the Hessian is calculated.

$$Trace[H_\Lambda] \,=\, \Delta\Lambda \qquad (15)$$

Example 1.1. *The empty regions.*

I would like to recall at this stage that, in classical physics, the empty regions of our universe are often characterized by the presence of fields (electric and gravitational) of which the Laplace operator vanishes. Hence, in that theory, these classical regions are characterized by a set of Hessian matrices, the trace of which vanishes and by an equation of motion such that:

$$\{Equ.(14),\, \Delta\Lambda \,=\, 0\}$$

$$\Downarrow$$

$$\frac{d...}{dt}_{vacuum} \equiv -[H_\Lambda] \tag{16}$$

$$|_\Lambda \mathbf{X} >= -[H_\Lambda] \,.\, |_\Lambda \mathbf{s} > \tag{17}$$

And:

$$\{_{[J]}\Phi(_\Lambda \mathbf{s}),\, [H_\Lambda]\} = \frac{d_{[J]}\Phi(_\Lambda \mathbf{s})}{dt} = {}_{[J]}\Phi(-[H_\Lambda] \,.\, |_\Lambda \mathbf{s} >) \tag{18}$$

With this mathematical condition, the preservation of the singular vector is obtained when and if:

$$[H_\Lambda] \,.\, |_\Lambda \mathbf{s} >= |\mathbf{0} > \tag{19}$$

This relation represents a strong constraint on the formalism of the Hessian matrix. One of them is that, for a non-vanishing singular vector, that Hessian matrix has a vanishing determinant. Since that matrix is related to a polynomial Λ appearing with a decomposition, that polynomial is degenerated and the discussion must introduce a type II Kern; see Remark1.2.

Remark 1.8. *The commutator [Φ, H].*

Let calculate the bracket [Φ, H] resulting from the calculations that have been made in Remark1.5, Equ.(5).

$$[_{[J]}\Phi(_\Lambda \mathbf{s}),\, [H_\Lambda]]$$

$$=$$

$$_{[J]}\Phi(_\Lambda \mathbf{s}) \,.\, [H_\Lambda] - [H_\Lambda] \,.\, {}_{[J]}\Phi(_\Lambda \mathbf{s})$$

$$=$$

$$Equ(8) - Equ.(7)$$

$$=$$

$$\begin{bmatrix} H_{31} . s^2 - H_{21} . s^3 & H_{32} . s^2 - H_{22} . s^3 & H_{33} . s^2 - H_{23} . s^3 \\ H_{11} . s^3 - H_{31} . s^1 & H_{12} . s^3 - H_{32} . s^1 & H_{13} . s^3 - H_{33} . s^1 \\ H_{21} . s^1 - H_{11} . s^2 & H_{22} . s^1 - H_{12} . s^2 & H_{23} . s^1 - H_{13} . s^2 \end{bmatrix}$$

$$-$$

$$\begin{bmatrix} H_{12} . s^3 - H_{13} . s^2 & H_{13} . s^1 - H_{11} . s^3 & H_{11} . s^2 - H_{12} . s^1 \\ H_{22} . s^3 - H_{23} . s^2 & H_{23} . s^1 - H_{21} . s^3 & H_{21} . s^2 - H_{22} . s^1 \\ H_{32} . s^3 - H_{33} . s^2 & H_{33} . s^1 - H_{31} . s^3 & H_{31} . s^2 - H_{32} . s^1 \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} 2 . (H_{13} . s^2 - H_{12} . s^3) & H_{23} . s^2 - H_{13} . s^1 + (H_{11} - H_{22}) . s^3 & (H_{33} - H_{11}) . s^2 + (H_{12} . s^1 - H_{23} . s^3) \\ ... & 2 . (H_{12} . s^3 - H_{23} . s^1) & (H_{22} - H_{33}) . s^1 + (H_{13} . s^3 - H_{12} . s^2) \\ ... & ... & 2 . (H_{23} . s^1 - H_{13} . s^2) \end{bmatrix} \tag{20}$$

This matrix is symmetric. In opposition to what concerns the anti-commutator, it is not easy to start something with. Let remark that any symmetric Hessian matrix can be understood as a bi-vector $(\mathbf{H}_{in}, \mathbf{H}_{off})$ in E(3, C) × E(3, C), the components of which are per convention disposed like this:

$$[H_\Lambda] = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{12} & H_{22} & H_{23} \\ H_{13} & H_{23} & H_{33} \end{bmatrix} = \begin{bmatrix} H^1_{in} & H^3_{off} & H^2_{off} \\ H^3_{off} & H^2_{in} & H^1_{off} \\ H^2_{off} & H^1_{off} & H^3_{in} \end{bmatrix} \tag{21}$$

With this convention, it is possible to state that:

$$[{}_{[J]}\Phi({}_{\Lambda}\mathbf{s}),\, [H_{\Lambda}]]$$
$$=$$
$$\begin{bmatrix} 2\,.\,(H_{13}\,.\,s^2 \;-\; H_{12}\,.\,s^3) & (\mathbf{H}_{off} \wedge {}_{\Lambda}\mathbf{s})^3 \;+\; (H_{11} \;-\; H_{22})\,.\,s^3 & (\mathbf{H}_{off} \wedge {}_{\Lambda}\mathbf{s})^2 \;+\; (H_{33} \;-\; H_{11})\,.\,s^2 \\ ... & 2\,.\,(H_{12}\,.\,s^3 \;-\; H_{23}\,.\,s^1) & (\mathbf{H}_{off} \wedge {}_{\Lambda}\mathbf{s})^1 \;+\; (H_{22} \;-\; H_{33})\,.\,s^1 \\ ... & ... & 2\,.\,(H_{23}\,.\,s^1 \;-\; H_{13}\,.\,s^2) \end{bmatrix}$$

It would be great to be able to go further! This commutator will reappear later in Rem.(2.7) and play a strategic role.

Remark 1.9. *Calculating the square of a type I Kern.*

Let calculate the following generic quantity where a transposed Kern does not appear:

$$\forall\,|A| \;=\; \pm 1:\; K_{\Lambda_1}([P]_{|A|})\,.\,K_{\Lambda_2}([P]_{|A|}) \;+\; K_{\Lambda_2}([P]_{|A|})\,.\,K_{\Lambda_1}([P]_{|A|})$$

That quantity should vanish if a canonical anti-commutative relation involving a type I Kern is needed; see [[12]; p. 7, (2a.6)]. But this suggestion is a sore point because I must precise my canonical variables. Any way, with the data contained in this document, this generic quantity is:

$$\{\frac{1}{2}\,.\,[H_{\Lambda_1}] \;-\; |A|\,.\,{}_{[J]}\Phi({}_{\Lambda_1}\mathbf{s})\}\,.\,\{\frac{1}{2}\,.\,[H_{\Lambda_2}] \;-\; |A|\,.\,{}_{[J]}\Phi({}_{\Lambda_2}\mathbf{s})\}$$
$$+\{\frac{1}{2}\,.\,[H_{\Lambda_2}] \;-\; |A|\,.\,{}_{[J]}\Phi({}_{\Lambda_2}\mathbf{s})\}\,.\,\{\frac{1}{2}\,.\,[H_{\Lambda_1}] \;-\; |A|\,.\,{}_{[J]}\Phi({}_{\Lambda_1}\mathbf{s})\}$$
$$=$$
$$\frac{1}{4}\,.\,\{[H_{\Lambda_1}]\,.\,[H_{\Lambda_2}] \;+\; [H_{\Lambda_2}]\,.\,[H_{\Lambda_1}]\} \;+\; \{{}_{[J]}\Phi({}_{\Lambda_1}\mathbf{s})\,.\,{}_{[J]}\Phi({}_{\Lambda_2}\mathbf{s}) \;+\; {}_{[J]}\Phi({}_{\Lambda_2}\mathbf{s})\,.\,{}_{[J]}\Phi({}_{\Lambda_1}\mathbf{s})\}$$
$$-\,\frac{|A|}{2}\,.\,\{[H_{\Lambda_1}]\,.\,{}_{[J]}\Phi({}_{\Lambda_2}\mathbf{s}) \;+\; {}_{[J]}\Phi({}_{\Lambda_1}\mathbf{s})\,.\,[H_{\Lambda_2}] \;+\; [H_{\Lambda_2}]\,.\,{}_{[J]}\Phi({}_{\Lambda_1}\mathbf{s}) \;+\; {}_{[J]}\Phi({}_{\Lambda_2}\mathbf{s})\,.\,[H_{\Lambda_1}]\}$$

In peculiar, for a unique given Lambda function ($\Lambda_1 = \Lambda_2 = \Lambda$), this quantity is:

$$\forall\,|A| \;=\; \pm 1: \tag{22}$$
$$2\,.\,K^2_{\Lambda}([P]_{|A|})$$
$$=$$
$$\frac{1}{2}\,.\,[H_{\Lambda}]^2 \;+\; 2.\,{}_{[J]}\Phi^2({}_{\Lambda}\mathbf{s}) \;-\; |A|\,.\,\{[H_{\Lambda}]\,.\,{}_{[J]}\Phi({}_{\Lambda}\mathbf{s}) \;+\; {}_{[J]}\Phi({}_{\Lambda}\mathbf{s})\,.\,[H_{\Lambda}]\}$$
$$=$$
$$\frac{1}{2}\,.\,[H_{\Lambda}]^2 \;+\; 2.\,{}_{[J]}\Phi^2({}_{\Lambda}\mathbf{s}) \;-\; |A|\,.\,{}_{[J]}\Phi(\{Tr[H_{\Lambda}]\,.\,Id_3 \;-\; [H_{\Lambda}]\}\,.\,|{}_{\Lambda}\mathbf{s}>)$$

Remark 1.10. *Conditions insuring the vanishing of the anti-symmetric part.*

Let now look for the conditions which would insure the nullity of the anti-symmetric part of the square of a given type I Kern.

$${}_{[J]}\Phi({}_{\Lambda}\mathbf{X}) \;=\; [0]_3 \iff {}_{\Lambda}\mathbf{X} = \mathbf{0} \tag{23}$$

Two evident opportunities appears:

- either the singular vector vanishes: ${}_{\Lambda}\mathbf{s} = \mathbf{0}$;

- or, whatever the singular vector is:

$$\forall\, {}_{\Lambda}\mathbf{s} \neq \mathbf{0} : \{Tr[H_{\Lambda}] \,.\, Id_3 \,-\, [H_{\Lambda}]\} \,.\, |_{\Lambda}\mathbf{s} >= |\mathbf{0} > \tag{24}$$

Beside the trivial solution which is realized when:

$$Tr[H_{\Lambda}] \,.\, Id_3 \,-\, [H_{\Lambda}] \,=\, [0] \tag{25}$$

I face a eigenvalue problem imposing the calculation and the systematic resolution of:

$$\begin{vmatrix} H_{11} - \Delta\Lambda & H_{12} & H_{13} \\ H_{12} & H_{22} - \Delta\Lambda & H_{23} \\ H_{13} & H_{23} & H_{33} - \Delta\Lambda \end{vmatrix} = 0 \tag{26}$$

Independently on the details of the calculations which will be done below, the anti-symmetric part in Equ.(22) vanishes when, simultaneously, (i) the Hessian matrix can be transformed into a diagonal matrix and (ii) its eigenvalues are coinciding with its trace.

Lemma 1.1. *Let consider any Hessian matrix and the quantity:*

$$T \,=\, (H_{11} \,.\, H_{22} \,+\, H_{22} \,.\, H_{33} \,+\, H_{33} \,.\, H_{11}) \,-\, ((H_{12})^2 \,+\, (H_{23})^2 \,+\, (H_{13})^2)$$

Provided that (i) that quantity and (ii) the determinant of the Hessian matrix at hand do not vanish, then that Hessian (i) is involved in a type I Kern and (ii) has only one never vanishing eigenvalue. Otherwise, if $T = 0$, the Hessian at hand is involved in a type II Kern.

Proof. : Let calculate the solutions of Equ.26; they are the solutions of the polynomial:

$$|H_{\Lambda}| \,-\, T \,.\, \Delta\Lambda \,+\, Tr[H_{\Lambda}] \,.\, (\Delta\Lambda)^2 \,-\, (\Delta\Lambda)^3 \,=\, 0$$

Here, exceptionally (recall Equ.15):

$$Tr[H_{\Lambda}] \,=\, \Delta\Lambda$$

This fact drastically reduces the polynomial:

$$|H_{\Lambda}| \,-\, T \,.\, \Delta\Lambda \,=\, 0 \tag{27}$$

Therefore, if:

$$T \,=\, 0$$

the determinant of the Hessian at hand vanishes. As consequence, that Hessian is involved in a type II Kern. Otherwise if:

$$T \,\neq\, 0 \tag{28}$$

and if the Hessian is in the first category (i.e.: its determinant does not vanish; $|\mathrm{H}| \neq 0$), there is always a unique and never vanishing solution:

$$\Delta\Lambda(\mathbf{a}) \tag{29}$$

$$=$$

$$\frac{\partial^2\Lambda(\mathbf{a})}{\partial^2 a^1} \,+\, \frac{\partial^2\Lambda(\mathbf{a})}{\partial^2 a^2} \,+\, \frac{\partial^2\Lambda(\mathbf{a})}{\partial^2 a^3}$$

$$=$$

$$\frac{|H_{\Lambda}|}{(H_{11} \,.\, H_{22} \,+\, H_{22} \,.\, H_{33} \,+\, H_{33} \,.\, H_{11}) \,-\, ((H_{12})^2 \,+\, (H_{23})^2 \,+\, (H_{13})^2)} \neq 0$$

□

1.3 A pedagogical example.

Only for the pedagogy, let now look for the conditions that would allow the construction of a set of canonical anti-commutative relations when a Kern and its transposed are presumably forming a pair of independent variables.

1. Inspired by [[12]; p. 7, first equation in (2a.6) for the case f = g] and by [[13]; exercise 1.1, (1.22) and §1.2.2, A3, (1.32)], considering the Equ.(3) again, I bet that the first necessary condition has the following generic formalism where the r.h.t. is a kind of norm for the polynomial Λ at hand; at this stage, that norm has yet to be defined more precisely:

$$(a) : \{\downarrow(\Lambda), \uparrow(\Lambda)\} \; = \; \frac{1}{2} \, . \{[H_\Lambda]^2 \; - \; {}_{[J]}\Phi^2({}_\Lambda\mathbf{s})\} \; = < \Lambda, \Lambda > \, . Id_3 \tag{30}$$

   This relation means in peculiar that a Kern anti-commutes with its transposed if and when the norm of the polynomial at hand vanishes.

2. The second necessary condition is realized when the Equ.(22) vanishes:

$$(b) : \frac{1}{2} \, . [H_\Lambda]^2 \; + \; 2. {}_{[J]}\Phi^2({}_\Lambda\mathbf{s}) \; - \; |A| \, . {}_{[J]}\Phi({}_\Lambda\mathbf{X}) \; = \; [0]_3 \tag{31}$$

   Due to its own formalism, that second necessary condition implicitly contains two sub-conditions (one for its symmetric part and another one for its anti-symmetric part) that must be realized simultaneously; more exactly:

$$(b_1) : \frac{1}{2} \, . [H_\Lambda]^2 \; + \; 2. {}_{[J]}\Phi^2({}_\Lambda\mathbf{s}) \; = \; [0]_3 \tag{32}$$

   and the already known Equ.(24), recall:

$$(b_2) : \forall \, |A| \; = \; \pm 1 : \; \{Tr[H_\Lambda] \, . Id_3 \; - \; [H_\Lambda]\} \, . |{}_\Lambda\mathbf{s} > \; = |\mathbf{0} >$$

Remark 1.11. *Important discussion.*

The proposed axiomatic is characterized by the simultaneous validity of Equ.(a), (b_1) and (24). But, let also remark that:

- When (a) and (b_1) are realized simultaneously, then it is really easy to get two new relations; precisely:

$$4 \times (a) \; + \; (b-1) \; \Rightarrow \; [H_\Lambda]^2 \; = \; \frac{8}{5} \, . < \Lambda, \Lambda > \, . Id_3 \tag{33}$$

  and:

$$(b-1) \; - \; (a) \; \Rightarrow \; {}_{[J]}\Phi^2({}_\Lambda\mathbf{s}) \; = \; -\frac{2}{5} \, . < \Lambda, \Lambda > \, . Id_3 \tag{34}$$

  The new equation (33) is implicitly true in two cases:

$$(33-a) : [H_\Lambda] \; - \; \sqrt{\frac{8}{5}} \, . < \Lambda, \Lambda >^{1/2} \, . Id_3 \; = \; {}^{(3)}[0]$$

$$(33-b) : [H_\Lambda] \; + \; \sqrt{\frac{8}{5}} \, . < \Lambda, \Lambda >^{1/2} \, . Id_3 \; = \; {}^{(3)}[0]$$

  Recalling the discussion concerning the anti-symmetric part of Equ.(4), let define the norm of the Λ function as:

$$(35-a) \; \Rightarrow \; (33-a) : \sqrt{\frac{8}{5}} \, . < \Lambda, \Lambda >^{1/2} = Tr[H_\Lambda] \tag{35}$$

$$(35-b)\ \rightarrow\ (33-b):\ \sqrt{\frac{8}{5}}\,.<\Lambda,\Lambda>^{1/2}=-Tr[H_\Lambda]$$

With that choice, the condition (24) is automatically true; hence there is the logical path:

$$\{(a),\,(b_1)\}\ \Rightarrow\ \{(33),\,(24)\};\ \{(33),\,(35)\}\ \Rightarrow\ (24)$$

The necessary conditions (a), (b_1) and (24) can be replaced by the conditions (a), (b_1) and (35), the bonus of which being the definition of a norm for the polynomial Λ at hand.

- When (i) the plausible norm for the Λ function is given either with (35-a) or (35-b) and (ii) the condition (24) is supposed to be true, then the Equ.(33) is automatically true too.

- Furthermore, in injecting (33) into (a), I now recover (34). Since (33) and (34) are now simultaneously true, it is the same for (a) and (b_1). When the condition (b_1) is true, then the conditions (31) and (24) as well are true too. Hence, the Equ.(35), (24) and (30) are sufficient conditions for a recovery of (31).

$$\{(35),\,(24)\}\ \Rightarrow\ (33)$$
$$\{(33),\,(30)\}\ \Rightarrow\ (34)$$
$$\{(33),\,(34)\}\ \Rightarrow\ \{(30),\,(32)\}\ \rightarrow\ \{(30),\,(31),\,(34)\}$$

- when (34) is injected into (30), then (33) is recovered.

$$\{(30),\,(34)\}\ \rightarrow\ (33)$$

Observing attentively the three previous logical chains, I conclude that the Equ.(30), (33) and (35) are sufficient conditions for the construction of canonical anti-commutative relations with any type I Kern. They are insuring a plausible similitude between the theory which is exposed here and the one which is allowing the construction of a set of canonical anti-commutative relations (like, e.g.: in [[12]; p. 7, (2a.6)]) .

Theorem 1.1. *Canonical anti-commutative relations for the theory of the (E) Question.*

Let consider any non-degenerated polynomial of degree two, Λ. That polynomial can implicitly be associated with the pair ($[H_\Lambda]$, ${}_{[J]}\Phi({}_\Lambda\mathbf{s})$) in M(3, C) × M(3, C). That pair contains the necessary ingredients allowing the construction of another pair ([K], $[K]^t$) in M(3, C) × M(3, C) too. That pair is (i) a pair of non-trivial solutions for the (E) question; see [[a]] and (ii) a pair of operators (one for the creation and the other one for the annihilation) when the conditions (30), (33) and (35) are true.

- The first canonical relation is:

$$\forall\,\Lambda\,\in\,Pol(2)E(3,\,C)^*,\,\forall\,\downarrow=\,[K_\Lambda],\,\uparrow=\,[K_\Lambda]^t\,\in\,M(3,\,C):\qquad(36)$$
$$\downarrow\,.\,\uparrow\,+\,\uparrow\,.\,\downarrow=\frac{1}{2}.\{[H_\Lambda]^2\ -\ {}_{[J]}\Phi^2({}_\Lambda\mathbf{s})\}\ =\ \frac{5}{8}.Tr[H_\Lambda]^2.Id_3$$

Furthermore, due to the information contained in lemma 1.1, when the Laplace operator of the Λ polynomial does not vanish, that relation can be rewritten for a pair of "normalized operators":

$$(\frac{[K_\Lambda]}{Tr[H_\Lambda]},\,\frac{[K_\Lambda^t]}{Tr[H_\Lambda]})$$

- The second canonical relation is signing a self-destructive interaction:

$$\downarrow . \downarrow = [K_\Lambda]^2 = {}^{(3)}[0] \tag{37}$$

  I remark by the way that any type I Kern respecting that second relation can be related to the concept of pure spinor; see [[07]; §106].

2 Building a coherent set of canonical anti-commutative relations.

2.1 Looking for the representation of the Hamiltonian.

Remark 2.1. *A remaining question.*

One crucial and logical question remains unanswered: "Is the pair ([K], $[\mathrm{K}]^t$) a pair of canonical conjugated variables?" To have a chance to find an answer, A Hamiltonian must be introduced somewhere in that discussion. Since the Hamiltonian related to an angular momentum does not spontaneously appear in the previous discussion, I shall loan a little bit to the scientific literature on that topic.

Remark 2.2. *Similitude with the academic formulation.*

Here, I am implicitly working with:

$$a^*(\Lambda) = \downarrow = K_\Lambda([P]_{|A|}) = |A| . \{\frac{1}{2} . [H_\Lambda] - |A| . {}_{[J]}\Phi({}_\Lambda\mathbf{s})\}$$

$$a(\Lambda) = \uparrow = K^t_\Lambda([P]_{|A|}) = |A| . \{\frac{1}{2} . [H_\Lambda] + |A| . {}_{[J]}\Phi({}_\Lambda\mathbf{s})\}$$

Furthermore:

$$a(\Lambda) + a^*(\Lambda) = |A| . [H_\Lambda]$$

$$a(\Lambda) - a^*(\Lambda) = {}_{[J]}\Phi(2 . {}_\Lambda\mathbf{s})$$

The second relation indicates that the difference between both operators is an image of twice the singular vector. This viewpoint suggests that a Hamiltonian may eventually be associated with the product:

$$a(\Lambda) . a^*(\Lambda)$$

$$=$$

$$\uparrow . \downarrow$$

$$=$$

$$\{\frac{1}{2} . [H_\Lambda] + |A| . {}_{[J]}\Phi({}_\Lambda\mathbf{s})\} . \{\frac{1}{2} . [H_\Lambda] - |A| . {}_{[J]}\Phi({}_\Lambda\mathbf{s})\}$$

$$=$$

$$\frac{1}{4} . [H_\Lambda]^2 - {}_{[J]}\Phi^2({}_\Lambda\mathbf{s}) + \frac{|A|}{2} . \{[H_\Lambda] . {}_{[J]}\Phi({}_\Lambda\mathbf{s}) - {}_{[J]}\Phi({}_\Lambda\mathbf{s}) . [H_\Lambda]\}$$

$$=$$

$$\frac{1}{4} . [H_\Lambda]^2 - {}_{[J]}\Phi^2({}_\Lambda\mathbf{s}) - \frac{|A|}{2} . [{}_{[J]}\Phi({}_\Lambda\mathbf{s}), [H_\Lambda]]$$

The third term in the sum has been calculated in the Rem(1.8); see Equ.(20). It is a symmetric matrix and the whole product ↑.↓ too. It does not yet mean that that product is representing a Hamiltonian operator.

Remark 2.3. *Another approach: Einstein's theory of relativity.*

There is another possibility to introduce a mathematical object representing the energy; it is the approach involving Einstein's theory of relativity. In the second part of that exploration, I have proposed three fundamental identifications. One of them involves the inverse of the spatial metric [Part II; remark 1.6, p.5, (16)]. Since that exploration concerns regions of the universe such that ($R_{\lambda\mu} = 0$), and since Einstein's theory is supposed to be valid in these regions [[08]; p.329, (95,8)] :

$$T_{\lambda\mu} = \frac{T}{2} \cdot g_{\lambda\mu}$$

The determinant of the inverse spatial metric has been calculated in [Part II; remark 1.11]. When it does not vanish, the spatial metric can be calculated and, as consequence of Einstein's theory, the spatial part of the energy-impulse tensor too. This affirmation holds at the level of principles. But, what is the logical link between a Kern resulting from a decomposition and the matrix representing the inverse of the spatial metric? The answer lies in the document [[b]; (1.10) and (1.13)]. The junction is done through the dispersion relation and the inverse spatial metric is itself a Kern!

2.2 Escapade in the fourth dimension.

Proposition 2.1. *There are deformed tensor products in Einstein' theory.*

Proof. : Newtonian gravitational fields represent the limit cases in Einstein' theory of gravity [[14]; chapter 17, §4, p. 415], [[15]; §1.11, pp. 50-51, especially (1.11.11)]; that theory proposes an equation describing the deviation of geodesics (EDG):

$$\delta \ddot{x^\alpha} = -R^\alpha{}_{0\beta 0} \,.\, \delta x^\beta = \frac{\partial^2 \phi}{\partial x^\alpha \partial x^\beta} \,.\, \delta x^\beta \tag{38}$$

where ϕ is a gravitational potential which, traditionally, depends on $\mathbf{x}$: (c.t, x, y, z) = (x^0, x^1, x^2, x^3). Let focus the attention on a territory where the metric is reversible; the tensor calculus automatically allows:

$$-R_{\omega 0\beta 0} = g_{\omega\alpha} \cdot \frac{\partial^2 \phi}{\partial x^\alpha \partial x^\beta} \tag{39}$$

The right hand term is the product of two matrices: the first one represents the metric [G] and the second one is a classical Hessian matrix acting on the gravitational potential. It is implicitly supposed that that potential depends on $\mathbf{x}$; hence, the EDG can be rewritten as:

$$-[R_{\omega 0\beta 0}] = ([G]^{-1})^{-1} \,.\, Hess_{(\mathbf{x},0)}\phi(\mathbf{x}) \tag{40}$$

In the territory where the metrics can be inverted, let consider the generic deformed tensor product:

$$\otimes_A(..., \mathbf{x})$$

Let also recall that, in opposition to the intrinsic method, the extrinsic method [[d]] can be involved without limitation for what concerns the cubes and the dimensions. That generic deformed tensor product can be decomposed with this method, provided that method satisfies a complementary logical test successfully. When it is the case, the dual representation of that kind of deformed tensor product accepts a non-trivial decomposition with the following formalism (see [[d]]; §1.4., p. 5]):

$$| \otimes_A (..., \mathbf{x}) >= \{{}_A\Phi(...) - \frac{1}{2} \,.\, [B]^{-1} \,.\, Hess_{(\mathbf{x},0)}\phi(\mathbf{x})\} \,.\, |\mathbf{x}> + |\mathbf{z}> \tag{41}$$

As a consequence, when [B] = $[G]^{-1}$, the equation (40) can be injected into the equation (41):

$$| \otimes_A (..., \mathbf{x}) >= \{_A\Phi(...) + \frac{1}{2} . [R_{\omega 0 \beta 0}]\} . |\mathbf{x} > + |\mathbf{z} > \quad (42)$$

In the subset where [B] = $[G]^{-1}$ and with that interpretation, the half of the matrix $[R_{\omega 0 \beta 0}]$ represents a difference between a non-trivial decomposition and a trivial one for the set of $\otimes_A(\mathbf{...}, \mathbf{x})$ deformed tensor products. Due to specific conditions imposed by the extrinsic method (see [[d]; §1.4., p. 5] again), this proposition holds only true for weak potentials; concretely, when: $\phi(\mathbf{x}) = 0(3)$. That condition is in coincidence with the one which is imposed by the physical circumstances ($\phi \ll c^2$) concerning the weak gravitational fields limit. □

Lemma 2.1. *The Newtonian limit.*

The deformed tensor products can be introduced into Einstein's theory of gravity at the Newtonian limit through the equation describing the deviations of geodesics.

Remark 2.4. *Comments on the lemma 2.1.*

That conclusion gives rise to at least two important questions:

- Q1: What are the correct physical interpretations for the generic cube A and the generic vector $^{(4)}\mathbf{...}$? It is obvious that I am interested by the double eventuality: (i) the unidentified vector $^{(4)}\mathbf{...}$ represents a speed $^{(4)}\mathbf{u}$ and, simultaneously, (ii) the cube A is anti-symmetric; the effective realization of that case focuses the discussion on deformed angular momentum acting in a four-dimensional space.
- Q2: Provided the double eventuality is effectively realized, then what is the link, if there is any one, between the generic equation:

$$|[\mathbf{u}, \mathbf{x}]_A >= \{_A\Phi(\mathbf{u}) + \frac{1}{2} . [R_{\omega 0 \beta 0}]\} . |\mathbf{x} > + |\mathbf{z} > \quad (43)$$

and the Lorentz-Einstein Law (alias: the co-variant version of the Lorentz law in electromagnetism)? Recall that that law I can be written in R^4 [[14];chapter 1, p. 35, (1.11)] as:

$$m . |\frac{d\mathbf{u}}{ds} + \otimes_{\Gamma(2)}(\mathbf{u}, \mathbf{u}) >= q . [F^{\alpha}{}_{\beta}] . |\mathbf{u} > \quad (44)$$

Remark 2.5. *Introducing the Lagrange equations.*

Let write in a normalized style:

$$\mathbf{x} = \mathbf{q} \to \mathbf{q}_1 ; \mathbf{u} = \dot{\mathbf{q}} \to \mathbf{q}_2 ; \Rightarrow \mathbf{q}_2 = \frac{d\mathbf{q}_1}{d\lambda} \quad (45)$$

In any four dimensional space, Lagrange equations are:

$$\forall \alpha = 0, 1, 2, 3 : \frac{\partial L(\mathbf{q}_1, \mathbf{q}_2)}{\partial q_1^{\alpha}} - \frac{d}{d\lambda} \frac{\partial L(\mathbf{q}_1, \mathbf{q}_2)}{\partial q_2^{\alpha}} = 0 \quad (46)$$

Since, in general:

$$\forall f(\mathbf{q}_2) : \frac{df(\mathbf{q}_2)}{d\lambda} = \sum_{\beta=0}^{\beta=3} \frac{\partial f(\mathbf{q}_2)}{\partial q_2^{\beta}} . \frac{dq_2^{\beta}}{d\lambda} + 0(2) \quad (47)$$

The four peculiar following functions:

$$\forall \alpha = 0,1,2,3 : f_\alpha(\mathbf{q}_2) = \frac{\partial L(\mathbf{q}_1, \mathbf{q}_2)}{\partial q_2^\alpha} \tag{48}$$

are such that:

$$\forall \alpha = 0,1,2,3 : \frac{d}{d\lambda}\frac{\partial L(\mathbf{q}_1, \mathbf{q}_2)}{\partial q_2^\alpha} = \sum_{\beta=0}^{\beta=3} \frac{\partial \frac{\partial L(\mathbf{q}_1, \mathbf{q}_2)}{\partial q_2^\alpha}}{\partial q_2^\beta} . \frac{dq_2^\beta}{d\lambda} = \sum_{\beta=0}^{\beta=3} \frac{\partial^2 L(\mathbf{q}_1, \mathbf{q}_2)}{\partial q_2^\alpha \partial q_2^\beta} . \frac{dq_2^\beta}{d\lambda} \tag{49}$$

But, here especially:

$$\mathbf{q}_2 = \frac{d\mathbf{q}_1}{d\lambda} \Rightarrow \frac{d\mathbf{q}_2}{d\lambda} = \frac{d^2\mathbf{q}_1}{d^2\lambda} \tag{50}$$

The Lagrange equations furnish:

$$\forall \alpha = 0,1,2,3 : \frac{\partial L(\mathbf{q}_1, \mathbf{q}_2)}{\partial q_1^\alpha} - \sum_{\beta=0}^{\beta=3} \frac{\partial^2 L(\mathbf{q}_1, \mathbf{q}_2)}{\partial q_2^\alpha \partial q_2^\beta} . \frac{d^2 q_1^\beta}{d^2\lambda} = 0 \tag{51}$$

And, in fine:

$$|\mathbf{grad_x} L(\mathbf{x}, \mathbf{u}) > - Hess_\mathbf{u} L(\mathbf{x}, \mathbf{u}) . |\frac{d^2\mathbf{x}}{d^2\lambda} >= 0 \tag{52}$$

This formulation will be useful in the next remark.

Remark 2.6. *Interpretation in terms of elasticity.*

Within the Newtonian context, the spatial acceleration field due to a mass M at Euclidean distance R is related to the spatial position ${}^{(3)}\mathbf{x}$ in that way; $G =$ 6,67.10^{-11} MKSA:

$$\frac{d^2\mathbf{x}}{d^2t} = -\frac{G.M}{R^3} . {}^{(3)}\mathbf{x}$$

Let extrapolate this usual situation in supposing that there exists a matrix ${}^{(4)}$[M] such that the acceleration is related to the positioning in space-time in the following way:

$$|\frac{d^2\mathbf{x}}{d^2\lambda} >= {}^{(4)}[M] . |\mathbf{x} >, \tag{53}$$

This is transforming the Lagrange equations (52) into:

$$|\mathbf{grad_x} L(\mathbf{x}, \mathbf{u}) >= Hess_\mathbf{u} L(\mathbf{x}, \mathbf{u}) . [M] . |\mathbf{x} > \tag{54}$$

Let suppose that the double eventuality which has been sketched in Rem.(2.4 ; Q1) is realized. The discussion focuses now on a generalization of the deformed angular momentum. Let decompose $[{}^{(4)}\mathbf{x}, {}^{(4)}\mathbf{u}]_A$ with the help of the extrinsic method and get:

$$[P] - {}_A\Phi(\mathbf{x}) = -\frac{1}{2} . [B]^{-1} . [HessP_2(\mathbf{u})] \tag{55}$$

Let multiply this relation on both sides by ${}^{(4)}\mathbf{x}$:

$$\{[P] - {}_A\Phi(\mathbf{x})\} . |\mathbf{x} >= -\frac{1}{2} . [B]^{-1} . [HessP_2(\mathbf{u})] . |\mathbf{x} > \tag{56}$$

A formal similitude with the Equ.(54)) appears if I decide to write:

$$|\mathbf{grad_x} L(\mathbf{x}, \mathbf{u}) >= \{[P] - {}_A\Phi(\mathbf{x})\} . |\mathbf{x} > \tag{57}$$

and, for coherence:

$$\forall \mathbf{x} : \; Hess_{\mathbf{u}} L(\mathbf{x},\, \mathbf{u}) \,.\, [M] \;=\; -\frac{1}{2} \,.\, [B]^{-1} \,.\, [Hess P_2(\mathbf{u})] \tag{58}$$

Equ.(57) and (58) must be analyzed attentively. The first one proposes the existence of a Lagrange function, L, the values of which are depending on the degree of non-triviality in the decomposition of the deformed angular momentum $[\mathbf{x},\, \mathbf{u}]_A$. The second one resembles a gauge.

Anyway, when the double eventuality sketched in Rem.(2.4 ; Q1) is realized:

$$[\mathbf{u},\, \mathbf{x}]_A \,+\, [\mathbf{x},\, \mathbf{u}]_A \;=\; \mathbf{0}$$

In decomposing both representations of the deformed angular momentum simultaneously with the extrinsic method; recall the Equ.(43) and (55):

$$\{ {}_A\Phi(\mathbf{u}) \,+\, \frac{1}{2} \,.\, [R_{\omega 0 \beta 0}] \} \,.\, |\mathbf{x}> \,+ |\mathbf{z}>$$

$$+$$

$$\{ {}_A\Phi(\mathbf{x}) \,-\, \frac{1}{2} \,.\, [B]^{-1} \,.\, [Hess P_2(\mathbf{u})] \} \,.\, |\mathbf{u}> \,+ |\mathbf{Z}>$$

$$=$$

$$|\mathbf{0}>$$

Since the cube A is anti-symmetric (double eventuality):

$$\frac{1}{2} \,.\, [R_{\omega 0 \beta 0}] \,.\, |\mathbf{x}> \,+ |\mathbf{z}> \,-\frac{1}{2} \,.\, [B]^{-1} \,.\, [Hess P_2(\mathbf{u})] \,.\, |\mathbf{u}> \,+ |\mathbf{Z}> = \,|\mathbf{0}>$$

In accepting the identifications (57) and (58) when the double eventuality sketched in Rem.(2.4 ; Q1) is realized, I can roughly relate a part of the components of the Riemann-Christoffel curvature tensor to (i) the *kinetic* Hessian matrix of the Lagrangian L and (ii) to a (4-4) matrix [M] that can be suspected to contain important information on masses:

$$\{\frac{1}{2} \,.\, [R_{\omega 0 \beta 0}] \,.\, |\mathbf{x}> \,+ |\mathbf{z}>\} \,+\, \{Hess_{\mathbf{u}} L(\mathbf{x},\, \mathbf{u}) \,.\, [M] \,.\, |\mathbf{u}> \,+ |\mathbf{Z}>\} \;=\; |\mathbf{0}> \tag{59}$$

Unfortunately at this stage, this equation is not really useful. An important property of Einstein's theory of gravity is the implicit elasticity of the geometry. That characteristic is imperfectly translated here through the equation describing the deviations of geodesics, (43), the identifications (57) and (58) and the approximate synthesis (59).

2.3 A new derivation by respect for the time.

Remark 2.7. *Does a Kern and its transposed yes or not form a pair of conjugated operators for a theory of quantum mechanics?*

Coming back to the basics exposed in [[a]; corollary 3.1, p.19] it is easy to state that a Kern and its transposed are not totally independent because their sum is always equal to the Hessian matrix associated with the polynomial Λ resulting from a non-trivial decomposition.

A classical treatment of this topic requires that conjugated operators should be linked by canonical relations; for example in reading [[13]; after (1.15)], I learn that the condition:

$$|Hess_{\mathbf{u}} L(\mathbf{x},\, \mathbf{u})| \neq 0 \tag{60}$$

should be true if the Hamiltonian formalism must be discovered with the unique help of the Lagrangian at hand.

Remark 2.8. *Representing the evolution.*

My extrapolation here is asking if I can reasonably write the correspondences:

$$\mathbf{x} \equiv [K]\,? \tag{61}$$

$$\mathbf{u} \equiv [K]^t\,? \tag{62}$$

Even if the answer is positive, I cannot reach my goal as long as a system of derivations relating the evolution by respect for the time is missing in that intellectual construction. Fortunately, that item has already been addressed in Rem.(1.6 ; Equ.(14)). The translation of the very classical relation:

$$\frac{d\mathbf{x}}{dt} = \mathbf{u} \tag{63}$$

with the help of Equ.(14), forces to check the plausibility of:

$$\frac{d\mathbf{x}}{dt} \equiv \{Tr[H_\Lambda]\,.\,Id_3 \,-\, [H_\Lambda]\}\,.\,[K] \,=\, [K]^t \,\equiv\, \mathbf{u} \tag{64}$$

It is equivalent to explore:

$$\{Tr[H_\Lambda]\,.\,Id_3 \,-\, [H_\Lambda]\}\,.\,\{\frac{1}{2}\,.\,[H_\Lambda] \,-\, |A|\,.\,_{[J]}\Phi(_\Lambda\mathbf{s})\} \,=\, \{\frac{1}{2}\,.\,[H_\Lambda] \,+\, |A|\,.\,_{[J]}\Phi(_\Lambda\mathbf{s})\}$$

or:

$$\frac{1}{2}\,.\,[H_\Lambda]^2 + \frac{1}{2}\,.\,(1 - Tr[H_\Lambda])\,.\,[H_\Lambda] + |A|\,.\,(1 + Tr[H_\Lambda])\,.\,_{[J]}\Phi(_\Lambda\mathbf{s}) - |A|\,.\,[H_\Lambda]\,.\,_{[J]}\Phi(_\Lambda\mathbf{s}) \,=\, ^{(3)}[0]$$

The last term in that sum has been calculated in Equ.(6) and it can be any matrix; therefore the exploration must continue in separating this last term into two parts: its symmetric and its anti-symmetric one:

$$[H_\Lambda]\,.\,_{[J]}\Phi(_\Lambda\mathbf{s}) \,=\, \frac{1}{2}\,.\,\{[H_\Lambda],\, _{[J]}\Phi(_\Lambda\mathbf{s})\} \,+\, \frac{1}{2}\,.\,[[H_\Lambda],\, _{[J]}\Phi(_\Lambda\mathbf{s})]$$

Recall Equ.(12) and Rem.(1.8):

$$[H_\Lambda]\,.\,_{[J]}\Phi(_\Lambda\mathbf{s}) \,=\, \frac{1}{2}\,.\,_{[J]}\Phi(_\Lambda\mathbf{X}) \,+\, \frac{1}{2}\,.\,[[H_\Lambda],\, _{[J]}\Phi(_\Lambda\mathbf{s})] \tag{65}$$

Then state that the plausibility of the proposed relation is obtained when:

- for the anti-symmetric part:

$$\forall\,|A| \,=\, \pm 1 :\; (1 \,+\, Tr[H_\Lambda])\,.\,_{[J]}\Phi(_\Lambda\mathbf{s}) \,-\, \frac{1}{2}\,.\,_{[J]}\Phi(_\Lambda\mathbf{X}) \,=\, ^{(3)}[0]$$

$$\downarrow$$

$$(1 \,+\, Tr[H_\Lambda])\,.\,_\Lambda\mathbf{s} \,-\, \frac{1}{2}\,.\,_\Lambda\mathbf{X} \,=\, \mathbf{0} \tag{66}$$

  Recall the discussion in Rem.(1.11) and the manners to get the Equ.(23), or more concretely: the vanishing of $\mathbf{X}$; it is then easy to deduce the obligatory two possible conditions insuring the coherence of the proposed relation of derivation:

– (i) either the singular vector vanishes:

$$\mathbf{X} = \mathbf{0} \Rightarrow {}_{\Lambda}\mathbf{s} = \mathbf{0} \tag{67}$$

– or (ii) the trace of the Hessian matrix plus one vanishes:

$$\mathbf{X} = \mathbf{0} \Rightarrow 1 + Tr[H_{\Lambda}] = 0$$

Due to Equ.(15), that second eventuality can also be written as:

$$\Delta\Lambda + 1 = 0 \tag{68}$$

This surprising relation is equivalent to the condition: "The Laplace operator of the polynomial resulting from a given decomposition plus one is null". For the decomposition of angular momentum $[\mathbf{u},\, \mathbf{x}]_A$, there is a polynomial $\Lambda(\mathbf{u})$ and for the decomposition of the opposite deformed product $[\mathbf{x},\, \mathbf{u}]_A$, there is a polynomial $\Lambda(\mathbf{x})$. The formalism of the condition allowing the intervening of Equ.(14) is very interesting. Indeed, let imagine that the polynomials $\Lambda(\mathbf{x})$ and $\Lambda(\mathbf{u})$ act exactly like background potentials would do; they generate two vector fields (a classical and a kinetic gradient):

$$\mathbf{V} = \nabla_{\mathbf{x}}\Lambda(\mathbf{x}) \tag{69}$$
$$\mathbf{W} = \nabla_{\mathbf{u}}\Lambda(\mathbf{u})$$

Then, within a classical Euclidean context:

$$div\mathbf{V} = div(\nabla_{\mathbf{x}}\Lambda(\mathbf{x})) = \Delta_{\mathbf{x}}\Lambda(\mathbf{x}) \tag{70}$$
$$div\mathbf{W} = div(\nabla_{\mathbf{u}}\Lambda(\mathbf{u})) = \Delta_{\mathbf{u}}\Lambda(\mathbf{u})$$

If, furthermore, two equations of continuity exist ... something like:

$$div_{\mathbf{x}}\mathbf{V} = -\frac{\partial\Lambda(\mathbf{x})}{\partial t} \tag{71}$$
$$div_{\mathbf{u}}\mathbf{W} = -\frac{\partial\Lambda(\mathbf{u})}{\partial t}$$

We recover two generic relations:

$$\Delta_{\mathbf{x}}\Lambda(\mathbf{x}) + \frac{\partial\Lambda(\mathbf{x})}{\partial t} = 0 \tag{72}$$
$$\Delta_{\mathbf{u}}\Lambda(\mathbf{u}) + \frac{\partial\Lambda(\mathbf{u})}{\partial t} = 0$$

the formalism of which is resembling the one of the Equ.(68). At a first glance, there is no good reason justifying a coincidence between the polynomial $\Lambda(\mathbf{u})$ and a Newtonian potential of gravitation because in general:

$$\Delta_{\mathbf{u}}\Lambda(\mathbf{u}) \leftrightarrow \Delta_{\mathbf{x}}\phi_{grav}(\mathbf{x})?$$

But in a second analysis, the harmonization between the decomposition obtained through the help of the intrinsic method with the one which is reached through the intervening of the extrinsic method offers more eventualities; see [[e]; remark 2.9, pp. 13-15]. For example the first interpretation is allowing the coincidence of the Hessian matrices:

$$[Hess_{\mathbf{x}}\Lambda(\mathbf{x})] = [Hess_{\mathbf{u}}\Lambda(\mathbf{u})]$$

That coincidence has a simple and direct consequence on the respective Laplace operators (just calculate the trace of each Hessian matrix):

$$\Delta_{\mathbf{u}}\Lambda(\mathbf{u}) = \Delta_{\mathbf{x}}\Lambda(\mathbf{x}) \tag{73}$$

Hence, if in that context:

$$\Lambda(\mathbf{x}) = \phi_{grav.}(\mathbf{x}) \tag{74}$$

Then:

$$\Delta_{\mathbf{u}}\Lambda(\mathbf{u}) = \Delta_{\mathbf{x}}\phi_{grav}(\mathbf{x}) \tag{75}$$

- And, simultaneously for the symmetric part:

$$\frac{1}{2}\,.\,[H_\Lambda]^2 + \frac{1}{2}\,.\,(1 - Tr[H_\Lambda])\,.\,[H_\Lambda] + \frac{1}{2}\,.\,[[H_\Lambda],\,{}_{[J]}\Phi({}_\Lambda\mathbf{s})] = {}^{(3)}[0] \tag{76}$$

Once again, depending on how the validity of the condition concerning the anti-symmetric part has been obtained, there are two sub-cases:

  - (i) either the singular vector vanishes:

$$Equ.(67 \Rightarrow \{[H_\Lambda] + (1 - Tr[H_\Lambda])\,.\,Id_3\}\,.\,[H_\Lambda] = {}^{(3)}[0] \tag{77}$$

  - or (ii) the trace of the Hessian matrix plus one vanishes:

$$Equ.(68 \Rightarrow [H_\Lambda]^2 + [[H_\Lambda],\,{}_{[J]}\Phi({}_\Lambda\mathbf{s})] = {}^{(3)}[0] \tag{78}$$

Please note that if the square of the Hessian matrix vanishes, then the Hessian matrix commutes with the rotation matrix at hand.

Remark 2.9. *Introducing the gravitation into the discussion.*

Let go a step further along that vein and let explore:

$$\frac{d^2\mathbf{x}}{d^2t} = \frac{d\mathbf{u}}{dt} \equiv \{Tr[H_\Lambda]\,.\,Id_3 - [H_\Lambda]\}\,.\,[K]^t = -\frac{G\,.\,M}{R^3}\,.\,{}^{(3)}[K] \equiv -\frac{G\,.\,M}{R^3}\,.\,{}^{(3)}\mathbf{x} \tag{79}$$

There are several possibilities to study the plausibility of that relation; one of them is to check:

$$\{Tr[H_\Lambda]\,.\,Id_3 - [H_\Lambda]\}^2 + \frac{G\,.\,M}{R^3}\,.\,Id_3 = [0] \tag{80}$$

Since the first package must be true, I immediately consider both eventualities:

- The singular vector vanishes:

$$Equ.(67 \Rightarrow [H_\Lambda]^2 - 2\,.\,Tr[H_\Lambda]\,.\,[H_\Lambda] + (Tr^2[H_\Lambda] + \frac{G\,.\,M}{R^3})\,.\,Id_3 = [0]$$

There are two possibilities:

  - The Hessian matrix vanishes:

$$[H_\Lambda] = [0] \Rightarrow \{M = 0\,;\, R \to \infty\} \tag{81}$$

  - The Hessian matrix is proportional to the identity matrix:

$$[H_\Lambda] + (1 - Tr[H_\Lambda])\,.\,Id_3 = [0]$$

$$\Downarrow$$

$$(Tr[H_\Lambda] - 1)^2 - 2\,.\,Tr[H_\Lambda]\,.\,(Tr[H_\Lambda] - 1) + (Tr^2[H_\Lambda] + \frac{G\,.\,M}{R^3}) = 0$$

$$\Downarrow$$

$$1 + \frac{G\,.\,M}{R^3} = 0 \tag{82}$$

- The trace of the Hessian matrix at hand (equivalently, the Laplace operator of the polynomial Λ at hand) plus one vanishes:

$$Equ.(68 \Rightarrow \{Id_3 + [H_\Lambda]\}^2 + \frac{G\,.\,M}{R^3}\,.\,Id_3 = [0]$$

$$\downarrow$$

$$[H_\Lambda]^2 + 2\,.\,[H_\Lambda] + (1 + \frac{G\,.\,M}{R^3})\,.\,Id_3 = [0]$$

$$\downarrow \; Equ.(78)$$

$$[[H_\Lambda],\, {}_{[J]}\Phi({}_\Lambda\mathbf{s})] = 2\,.\,[H_\Lambda] + (1 + \frac{G\,.\,M}{R^3})\,.\,Id_3 \qquad (83)$$

That formulation exhibits the fact that, for homogeneous sources [2], the Laplace operator of the gravitational potential, therefore the density of matter of the source per unit volume too, is the indirect solution of an eigenvalue problem.

Remark that:

  - If, simultaneously, the singular vanishes, then either the Hessian matrix vanishes and, as obligatory consequence:

$$1 + \frac{G\,.\,M}{R^3} = 0$$

  or the former is true and I stay with a vanishing Hessian matrix:

$$[H_\Lambda] = [0]$$

Let recall that within a confrontation with the Klein-Gordon equation a non-vanishing singular vector is the signature for the presence of some Thirring-Lense like effect; see [Part II: Identifications, p. 5, p.17].

Example 2.1. *When the square of the Hessian matrix vanishes.*

Let suppose that the Equ.(68) is true and that the square of the Hessian matrix vanishes; then, the Hessian matrix commutes with the rotation matrix and I get:

$$[H_\Lambda]^2 = [0] \Rightarrow [H_\Lambda] + (\frac{1}{2} + \frac{G\,.\,M}{2\,.\,R^3})\,.\,Id_3 = [0] \qquad (84)$$

Or, for an homogeneous spherical source:

$$[H_\Lambda]^2 = [0] \Rightarrow [H_\Lambda] + (\frac{1}{2} + \frac{2\,.\,\pi}{3}\,.\,G\,.\,\rho)\,.\,Id_3 = [0] \qquad (85)$$

Please also note that, in that case:

$$[H_\Lambda]^2 = [0] \Rightarrow Trace\{[H_\Lambda] - \frac{1}{2}\,.\,Id_3\} = 2\,.\,\pi\,.\,G\,.\,\rho \qquad (86)$$

Up to a factor 2, this is the divergence of the Newtonian field.

[2] In that case the factor in front of the identity matrix is $4/3.\pi.\rho.G$ and that expression is directly related to the Laplace operator of the gravitational potential.

3 Abstract.

The initial aim of this document was to build a Hilbert space with matrices which are obtained through the decomposition of deformed cross products, especially of angular momentum.

To reach that goal, I have loaned some basics to the current axiomatic of quantum mechanics like they are for example exposed in [06], [10], [[12]] and [13]. The guiding idea is that (i) each decomposition is associated with a polynomial Λ which is entirely characterized by:

- a pair ([Hessian Λ], $_{\Lambda}\mathbf{s}$) for the non-degenerated cases;
- a bi-vector ($\mathbf{q}_1$, $\mathbf{q}_2$) and its associated pair (1/2.{$\mathrm{T}_2(\otimes)(\mathbf{q}_1, \mathbf{q}_2) + \mathrm{T}^t{}_2(\otimes)(\mathbf{q}_1, \mathbf{q}_2)$}, $_{[J]}\Phi(\mathbf{q}_1 \wedge \mathbf{q}_2)$) for the degenerated cases.

and that (ii) these characteristics should implicitly contain all necessary ingredients for a quantization.

This idea suggests that, *concerning the non-degenerated cases*, the pair ([K], $[\mathrm{K}]^t$) that can be formed with the previous one is a couple of conjugated operators allowing the construction of the first stones of a theory.

To prove the correctness of this idea, diverse brackets have been calculated. The anti-commutators {[K], $[\mathrm{K}]^t$} respect the canonical anti-commutative relations proposed in [[12]]. Furthermore, a derivation by respect for the time can be constructed with the ingredients which are contained in the pair ([Hessian Λ], $_{\Lambda}\mathbf{s}$); see Equ.(14).

That derivation has been tested to verify the plausibility of the equivalence between the classical pair ($\mathbf{x}$, $\mathbf{u}$) and the pair ([K], $[\mathrm{K}]^t$); see Rem.(2.8). The test brings essentially two types of admissible circumstances: (i) either the singular vector vanishes, Equ.(67), or (ii) the Laplace operator of the polynomial Λ at hand plus one is equal to zero, Equ.(68).

A confrontation with the identifications which have been proposed in the second part of that series indicates that the first type corresponds to a vanishing of the gyro-vector **g**, whilst the second type can be associated to some Thirring-Lense effect.

The temporary conclusion is that a non-degenerated polynomial is characterized by a pair ([K], $[\mathrm{K}]^t$), the arguments of which can be interpreted as conjugated operators in a theory of quantum mechanics. Furthermore, if the trace of the Hessian matrix that can be built in summing the arguments of that pair does not vanish, then these operators can be normalized; see the Theorem 1.1 and the Equ.(36) and (37).

The next step is to generalize this first approach with the intention to develop further a theory of quantum gravity. An interesting instrument to reach that goal is the discovery of a function (call it, f, if you want) allowing the definition of a correspondence between each Kern and a representation for it in E(3, C) via some (actually) unknown vector **U** in a way entirely loaned to E. Cartan logic exposed in [[07]; chapter III, p. 43]. Therefore, I envisage to generalize the relations proposed in [07] in writing:

$$K([P]_{|A|}) \in M(3,C) \xrightarrow{f} f(K([P]_{|A|})) = \mathbf{U} \in E(3,C)$$

$$|K([P]_{|A|})| \; = \; - < \mathbf{U}, \, \mathbf{U} >_{Id_3} \in \, C$$
$$K^2([P]_{|A|}) \; = < \mathbf{U}, \, \mathbf{U} >_{Id_3} \; . Id_3$$

1. Due to the first results which are accumulated in the theorem 1.1, the vector U must be a three-dimensional Euclidean isotropic vector; for more details concerning these vectors, please read [[07]]:

$$< \mathbf{U}, \, \mathbf{U} >_{Id_3} = \, 0$$

Recalling the proposed axiomatic, it is now obvious that that point imposes to work with any type I Kern having a vanishing determinant; as consequence:

$$< \mathbf{s} \, , \, \mathbf{s} >_{[H]} \; + \frac{|H|}{8} \; = \, 0 \, ; \, |H| \, \neq \, 0 \tag{87}$$

whilst there is no restriction for a type II Kern because its determinant is always null.

2. The anti (skew-) symmetric part of the square of the Kern must vanish. For a type I Kern, recall the discussion in Rem.(1.10).

3. The symmetric part of the square of the Kern must vanish.

Definition 3.1. *Pythagorean table.*

Let introduce the concept of Pythagorean table again, the definition of which is (for a better understanding of my notation):

$$(\mathbf{q}_1, \mathbf{q}_2) \in \, E(3, \, K) \times E(3, \, K) \; \xrightarrow{Pyth_{\otimes}} \; Pyth_{\otimes}(\mathbf{q}_1, \mathbf{q}_2) \; = \; T_2(\otimes)(< \mathbf{q}_1|, |\mathbf{q}_2 >)$$

with:

$$T_2(\otimes)(< \mathbf{q}_1|, |\mathbf{q}_2 >) \; = \; \begin{bmatrix} q_1^1 \cdot q_2^1 & q_1^2 \cdot q_2^1 & q_1^3 \cdot q_2^1 \\ q_1^1 \cdot q_2^2 & q_1^2 \cdot q_2^2 & q_1^3 \cdot q_2^2 \\ q_1^1 \cdot q_2^3 & q_1^2 \cdot q_2^3 & q_1^3 \cdot q_2^3 \end{bmatrix} \; \in \, M(3, \, K)$$

Remark 3.1. *Useful identities.*

A few number of remarkable identities must be recalled:

$$Py_{\otimes}^2(\mathbf{q}_1, \mathbf{q}_2) \; = < \mathbf{q}_1, \, \mathbf{q}_2 >_{Id_3} \; . Py_{\otimes}(\mathbf{q}_1, \mathbf{q}_2)$$
$$|Py_{\otimes}(\mathbf{q}_1, \mathbf{q}_2)| \; = \; 0$$
$${}_{[J]}\Phi^2(\mathbf{q}_1) \; = \; Py_{\otimes}(\mathbf{q}_1, \mathbf{q}_1) - \; < \mathbf{q}_1, \, \mathbf{q}_1 >_{Id_3} \; . Id_3$$

Remark 3.2. *The vanishing of the symmetric part.*

Once again, let consider the diverse possibilities

(a) **The type I Kern.**
Recall the discussion which has been developed in sub-section 1.3.

(b) **The type II Kerns**:
Due to the first remarkable relation concerning the Pythagorean matrices, the (mathematical) situation is a little bit simpler. Any pair of orthogonal vectors generates a Pythagorean table of which the square vanishes and any such table is a type II Kern.

At this stage, let come back to the discussion concerning the decomposition of deformed cross product at the Euclidean limit

$$\mathbf{q}_1 \wedge \mathbf{q}_2 \,=\, -\mathbf{q}_2 \,+\, \mathbf{z}\,;\, <\mathbf{q}_2,\, \mathbf{q}_2>_{Id_3} = \,0\,;\, \mathbf{z} \,\perp\, \mathbf{q}_2$$

and recall that the target $\mathbf{q}_2$ is orthogonal to the residual part $\mathbf{z}$. Hence for any such decomposition, this theory is able to build a type II Kern:

$$[K] \,=\, Py_{\otimes}(\mathbf{z}, \mathbf{q}_2)\,;\, [K]^2 \,=\, [0]$$

Its symmetric part plays the role of a Hessian for some unknown polynomial Ψ:

$$[K_+] \,=\, \frac{1}{2}\,.\,\{Py_{\otimes}(\mathbf{z}, \mathbf{q}_2) \,+\, Py^t_{\otimes}(\mathbf{z}, \mathbf{q}_2)\}$$

Its anti-symmetric part is (perhaps up to a minus sign):

$$[K_-] \,=\, \frac{1}{2}\,.\,\{Py_{\otimes}(\mathbf{z}, \mathbf{q}_2) \,-\, Py^t_{\otimes}(\mathbf{z}, \mathbf{q}_2)\} \,=\, {}_{[J]}\Phi(\mathbf{z} \wedge \mathbf{q}_2)$$

Hence the classical cross product between the target and the residual part plays a role equivalent to the one of a singular vector. As a matter of mathematical facts, these Kerns, their symmetric part and their anti-symmetric part as well have a vanishing determinant. The value $\mathrm{T}([\mathrm{K}_+])$ does not vanish because the target and the residual part are not parallel to each other:

$$T([K_+]) \,=\, \frac{1}{2}\,.\,(\mathbf{z} \wedge \mathbf{q}_2)^2$$

Hence we may formally associate a vanishing Laplace operator to the symmetric part of that Kern.

$$Trace([K_+]) \,=\, \frac{1}{2}\,.\,(\mathbf{z}\,.\,\mathbf{q}_2) \,=\, \Delta\Psi \,=\, 0$$

4 Acknowledgments.

Because I am an independent researcher, I am restricting myself to work with books related to the foundations of mathematics or of physics and to academic documents. Sometimes, I am obliged to loan some elements in the arXiv library. I warmly acknowledge the authors ... and the patience of my wife.

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