The Theory of the (E) Question

Does the new formalism of the EM field tensor contain a bi-vector "à la E. Cartan"?

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Content

The new formalism of the EM field tensor in continuous energetic contexts

Introduction

A previous exploration [a] has proposed a totally new logical path between the theory of relativity (A. Einstein's work; alias: GTR) and W. Heisenberg's uncertainty principle in a full four-dimensional context. The link is obtained with a specific treatment of the Lorentz-Einstein Law of motion involving a procedure allowing the decomposition of deformed tensor (resp. Lie) products. It yields a specific and new expression for the (2, 0) representation of the EM fields:

(01)

$$q. [F_{\mu\nu}] = ½. \{^{(4)}G. {}_{\blacktriangledown\Gamma}\Phi(^{(4)}\mathbf{p}) - {}_{\blacktriangledown\Gamma}\Phi^t(^{(4)}\mathbf{p}). {}^{(4)}G^t\}$$

Where q is the electrical charge of a particle with a 4D kinetic momentum $^{(4)}\mathbf{p}$ in some geometric environment characterized by the 4D metric $^{(4)}G$ and its variations; the latter are written in a condensed formalism which I have called the Levi-Civita or the Christoffel's cube $\blacktriangledown\Gamma$ containing the Christoffel's symbols of the second kind.

This expression is important because it contains terms depending on the local four-dimensional metric and on its variations. Just because of that fact, it can be suspected that EM fields have permanent interaction with the geometry. Hence, the new formulation should be able to give indications on the intensity of the interactions between both types of fields.

The aim of this document is to discover some consequences of that formalism.

Remark 01: General considerations

At a first glance, that new formalism (01) is directly related (i) to dynamical properties of the massive particle at hand and (ii) to the geometrical context (a priori any one). The dependence acts in such a way that a massless particle, a particle at rest or an invariant metric generates no EM field.

That means in peculiar that a massless and spin 2 graviton (the presumable carrier of the gravitational interaction) doesn't generate any EM field in a context where (01) holds true.

Conversely, the appearance of that kind of EM field is seemingly related to a dissymmetry which we may describe in saying that the product $^{(4)}G. {}_{\blacktriangledown\Gamma}\Phi(^{(4)}\mathbf{p})$ is not comparable with its transposed; for complementary calculation please see annex 01.

Concretely, the formalism (01) suggests the existence of real circumstances such that any non-vanishing pair ($\blacktriangledown\Gamma$, $^{(4)}\mathbf{p}$) - these circumstances can be described by the sentence: "The geometry is changing and the mass moves"- generates an EM field.

Remark 02: "How can the EM-potential vector be managed?"

The second remark is the total absence of the EM-potential vector, $^{(4)}\mathbf{A}$, in that expression (01).

From a classical viewpoint [01; p. 10, (46)] this is effectively totally surprising since that potential vector is a necessary mathematical tool to calculate the efficient EM field. This suggests, as it was and as it is very often the case in the literature, that the EM potential vector only is a tool without real existence. This doesn't mean that we cannot reintroduce it in the discussion. And this can in fact be done in different manners depending on the real circumstances. Just for the pedagogy, we could cite the retarded potentials [02, p. 193, (63,5)] approach and the Stükelberg-like trick consisting in a kind of translation: $\mathbf{p} \rightarrow \mathbf{p} - (q/c). \mathbf{A} = \pi$:

(02)

$$[F_{\mu\nu}]_{\mathbf{A}=\mathbf{0}} \rightarrow [F_{\mu\nu}]_{\mathbf{A}} = [F_{\mu\nu}]_{\mathbf{A}=\mathbf{0}} - \frac{q}{c}. \{G. {}_{\blacktriangledown\Gamma}\Phi(^{(4)}\mathbf{A}) - {}_{\blacktriangledown\Gamma}\Phi^t(^{(4)}\mathbf{A}). G^t\}$$

Nota bene: A potential is a static quantity: it must be attached to a set of topological points. Therefore, it is totally unrealistic to imagine that some EM-potential vector could eventually be parallel transported by respect for the local coordinate system. Furthermore, any motion is relative. The coordinate system can be arbitrarily chosen and can be different from a topological point to the next one.

To the legitimate question: "How do we have to manage the concept of EM-potential vector field?", a two steps answer can be done. First one may have a look at the theoretical considerations developed in [02; p. 208, (67.4)] relating a dipole and an EM-potential vector field; second one may consider the recent measurements of the cosmic background polarization [06: Planck satellite collaboration].

Remark 03: a constraint coming from A. Einstein's theory of relativity

The relation (01) should be confronted with another one (tensor calculus and partial derivations):

(03)

$$g_{\alpha\beta,\mu} = \Gamma_{\alpha\mu}{}^{\nu}.\, g_{\nu\beta} + \Gamma_{\beta\mu}{}^{\nu}.\, g_{\alpha\nu}$$

From which, even if the metric is not symmetric, it may easily be deduced (working on R or on C):

(04)

$$\forall\, p^{\mu}: g_{\alpha\beta,\mu}.\, p^{\mu} = \Gamma_{\alpha\mu}{}^{\nu}.\, g_{\nu\beta}.\, p^{\mu} + g_{\alpha\nu}.\, \Gamma_{\beta\mu}{}^{\nu}.\, p^{\mu} = \Gamma_{\alpha\mu}{}^{\nu}.\, p^{\mu}.\, g_{\nu\beta} + g_{\alpha\nu}.\, \Gamma_{\beta\mu}{}^{\nu}.\, p^{\mu}$$

Because the Levi-Civita connection is a symmetric one (see Christoffel's original work introducing the symbols: [15]), one can also write without loss of generality:

(05)

$$\forall\, p^{\mu}: g_{\alpha\beta,\mu}.\, p^{\mu} = (\Gamma_{\mu\alpha}{}^{\nu}.\, p^{\mu}).\, g_{\nu\beta} + g_{\alpha\nu}.\, (\Gamma_{\mu\beta}{}^{\nu}.\, p^{\mu})$$

Now let introduce colors to facilitate the progression and the understanding:

(06)

$$\forall\, p^{\mu}: g_{\alpha\beta,\mu}.\, p^{\mu} = (\Gamma_{\mu\alpha}{}^{\nu}.\, p^{\mu}).\, g_{\nu\beta} + g_{\alpha\nu}.\, (\Gamma_{\mu\beta}{}^{\nu}.\, p^{\mu})$$

Let also recall the following evidence in that theory:

(07)

$$\blacktriangledown_{\Gamma}\Phi({}^{(4)}\mathbf{p})_{\beta}{}^{\nu} = \Gamma_{\mu\beta}{}^{\nu}.\, p^{\mu}$$

It is now extremely easy to recognize that the second term of the sum in (06) is nothing but G. $\blacktriangledown_{\Gamma}\Phi({}^{(4)}\mathbf{p})$. And the first term of (06) is $\blacktriangledown_{\Gamma}\Phi^{t}({}^{(4)}\mathbf{p})$. G. Putting all together:

(08)

$$[\ldots\, g_{\alpha\beta,\mu}.\, p^{\mu} \ldots] = \blacktriangledown_{\Gamma}\Phi^{t}({}^{(4)}\mathbf{p}).\, G + G.\, \blacktriangledown_{\Gamma}\Phi({}^{(4)}\mathbf{p})$$

Let state that the l.h.t. of (08) is nothing but the ordinary derivative of the metric tensor by respect for some scalar (e.g.: s):

(09)

$$[\ldots\, g_{\alpha\beta,\mu}.\, p^{\mu} \ldots] = [\ldots \frac{\partial g_{\alpha\beta}}{\partial x^{\mu}}.\frac{dx^{\mu}}{ds} \ldots] = [\ldots \frac{dg_{\alpha\beta}}{ds} \ldots] = \frac{d}{ds} G = \dot{G}$$

In confronting (01) and (08) in the special cases of symmetric metrics:

(10-1)

$$G = G^{t}$$

(10-2)

$$2q.\, [F_{\mu\nu}] + \dot{G} = 2.\, G.\, \blacktriangledown_{\Gamma}\Phi({}^{(4)}\mathbf{p})$$

(10-3)

$$2q.\, [F_{\mu\nu}] - \dot{G} = -2.\, \blacktriangledown_{\Gamma}\Phi^{t}({}^{(4)}\mathbf{p}).\, G$$

The interesting point here is the fact that in absence of EM field:

(11)

$$\dot{G} = 2.\, \blacktriangledown_{\Gamma}\Phi^{t}({}^{(4)}\mathbf{p}).\, G$$

Nota bene: This relation is roughly evocating the formalism of the Ricci flow.

Remark 04: formal considerations on the formalism of the EM fields

Formal and esthetic considerations were in fact the first elements suggesting the existence of a G. $\blacktriangledown_{\Gamma}\Phi({}^{(4)}\ldots)$-like formalism for the (2, 0) EM field tensor representation. Indeed, observing the customized formalism of the (2, 0) tensorial representation of some EM field [02; p. 73], there is no difficulty in stating that the South-East corner of it is nothing but the trivial representation $\blacktriangledown_{\varepsilon}\Phi({}^{(3)}\mathbf{H})$ of some ${}^{(3)}\mathbf{H} \wedge {}^{(3)}\ldots$ unspecified cross product:

(12)

$$[F_{\alpha\beta}] = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -H_3 & H_2 \\ -E_2 & H_3 & 0 & -H_1 \\ -E_3 & -H_2 & H_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & <\mathbf{E}| \\ -|\mathbf{E}> & {}_{\varepsilon}\Phi(\mathbf{H}) \end{bmatrix}$$

Remark 05: EM fields and spinors

The relation (01) may also evoke a very precise situation for a mathematician; namely: E. Cartan's theory of spinors [08; chapter IX, p. 145, (1)].

The EM field predicted by the theory of the (E) question in a continuous context (this remark is related to the fact that the Hessian matrix appearing in the original formulation of the EM fields, see [a], disappears when the function is continuous) can eventualy represent an infinitesimal transformation of the metric G or of the trivial split $\blacktriangledown_\Gamma\Phi^t(^{(4)}\mathbf{p})$ if at least one of them is the representation of a bi-vector “à la E. Cartan”. This is exactly what will be now examine in all details.

Sub-remark 01: the trivial matrix cannot be an anti-symmetric matrix

Since the Levi-Civita cube is a symmetric cube, the following condition $\Gamma_{\delta\beta}{}^{\gamma} = - \Gamma_{\delta\gamma}{}^{\beta} = - \Gamma_{\gamma\delta}{}^{\beta} = \Gamma_{\gamma\beta}{}^{\delta} = \Gamma_{\beta\gamma}{}^{\delta} = - \Gamma_{\beta\delta}{}^{\gamma} = - \Gamma_{\delta\beta}{}^{\gamma}$ would impose a vanishing Levi-Civita cube for the coherence; which is a totally meaningless case here.

This sub-remark tells that the trivial matrix $\blacktriangledown_\Gamma\Phi^t(^{(4)}\mathbf{p})$ cannot be an anti-symmetric one; but it doesn't yet definitively disregard that matrix for the purpose one is scrutinizing (“Is sometimes the trivial matrix $\blacktriangledown_\Gamma\Phi^t(^{(4)}\mathbf{p})$ representing a bi-vector à la E. Cartan?”). I shall come back later on that item.

Sub-remark 02: a necessary condition of symmetry

In order to eventually be the representation of a bi-vector “à la E. Cartan”, the trivial matrix involved in (01) must be a symmetric matrix. Indeed, the formalism of [08; chapter IX, p. 145, (1)] is recovered when and if the metric and the trivial matrix involved in (01) are symmetric matrices.

Because $\Gamma_{\delta\beta}{}^{\gamma} = \Gamma_{\delta\gamma}{}^{\beta} = \Gamma_{\gamma\delta}{}^{\beta} = \Gamma_{\gamma\beta}{}^{\delta} = \Gamma_{\beta\gamma}{}^{\delta} = \Gamma_{\beta\delta}{}^{\gamma} = \Gamma_{\delta\beta}{}^{\gamma}$ is a coherent set of equalities, the eventual symmetry of the trivial matrix is totally compatible with the one of the Christoffel's symbols of the second kind. That eventuality just reduces the total number of independent Christoffel's symbols. The claim of the document is to verify if this mathematical opportunity can be effectively realized.

Sub-remark 03: metrics and bi-vectors

The purpose of that section

Let first explore if the metric tensor itself is representing a bi-vector “à la E. Cartan” [08]. In starting this purely algebraic investigation, I ignore if that eventuality is true in general (for any metric) but I know that empty space-times are supposedly associated with Minkowski geometry with signature (+ - - -) represented by the diverse matrices:

(13)

$$\eta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & <\mathbf{0}| \\ -|\mathbf{0}> & -\mathrm{Id}_3 \end{bmatrix} = \begin{bmatrix} -[K] & [0] \\ [0] & -[Id_2] \end{bmatrix}$$

Here [K] is the matrix introduced in [08; § 126, pp. 110-111]. And I also know that “allowed” metrics within the theory of relativity (GTR) are obtained in starting from η by successive deformations via the action of Jacobian matrices. So that the first quest is to look for matrices representing the Minkowski metric in the spirit explained in [08; p. 83, § 95]; that is: as if that metric would be the representation of a 2-vector.

The reader may argue that a lot of progresses have been made during the past century between the date of publication of [08] and today. That's true. For example, just following the logical set of accumulated knowledge: 1°) a metric tensor with signature (+ - - -), thus the one revealing Minkowski geometry, can always be constructed with the help of a Hermitian spinor [12; pp. 71-72]; 2°) the Cartan-Whittaker relation establishes a one to one correspondence between a spinor and a null bi-vector [18; p. 560], it can be induced that some Hermitian spinor are in a one to one correspondence with a null bi-vector. But this argument only means that that Hermitian spinor can help us to build a metric tensor in a way given in [12; pp. 71-72]. That way [12; p. 72, (2.3.2)] doesn't match the formalism proposed for bi-vectors in [08]. This is why I recommend the lecture of the section.

The algebraic investigation

If η would be such a bi-vector “à la E. Cartan”, then it would be defined by two vectors, themselves respectively represented by the matrices X_1 and X_2, and it would be itself represented by a matrix:

(14)

$$\eta = ½. \{X_1. X_2 - X_2. X_1\}$$

If I consider that the space-time is a 4D subspace of a E_5 space, then -following [08; § 93, p. 81, (5)]- I should write:

(15-i)

$$i = 1, 2: X_i = \begin{bmatrix} x_i^0 & x_i^1 & x_i^2 & 0 \\ x_i^{1\prime} & -x_i^0 & 0 & x_i^2 \\ x_i^{2\prime} & 0 & -x_i^0 & -x_i^1 \\ 0 & x_i^{2\prime} & -x_i^{1\prime} & x_i^0 \end{bmatrix}$$

Equipped with this formalism, I get a general representation for the r.h.t. of (14); indeed:

(16)

$$X_i. X_j$$

$$=$$

$$\begin{bmatrix} x_i^0.x_j^0 + x_i^1.x_j^{1'} + x_i^2.x_j^{2'} & x_i^0.x_j^1 - x_i^1.x_j^0 & x_i^0.x_j^2 - x_i^2.x_j^0 & x_i^1.x_j^2 - x_i^2.x_j^1 \\ x_i^{1'}.x_j^0 - x_i^0.x_j^{1'} & x_i^0.x_j^0 + x_i^{1'}.x_j^1 + x_i^2.x_j^{2'} & x_i^{1'}.x_j^2 - x_i^2.x_j^{1'} & -(x_i^0.x_j^2 - x_i^2.x_j^0) \\ x_i^{2'}.x_j^0 - x_i^0.x_j^{2'} & x_i^{2'}.x_j^1 - x_i^1.x_j^{2'} & x_i^0.x_j^0 + x_i^1.x_j^{1'} + x_i^{2'}.x_j^2 & x_i^0.x_j^1 - x_i^1.x_j^0 \\ x_i^{2'}.x_j^{1'} - x_i^{1'}.x_j^{2'} & -(x_i^{2'}.x_j^0 - x_i^0.x_j^{2'}) & x_i^{1'}.x_j^0 - x_i^0.x_j^{1'} & x_i^0.x_j^0 + x_i^{1'}.x_j^1 + x_i^{2'}.x_j^2 \end{bmatrix}$$

If I don't want to involve a 5D space, it is sufficient to refer to [08; § 125, pp. 129-130]. The formalism is now a simplification of the previous one. It is obtained in writing $x_i^0 = 0$.

(17)

$$(X_i.\ X_j) \text{ if } x_i^0 = 0 \text{ for } i = 1, 2$$

$$=$$

$$\begin{bmatrix} x_i^1.x_j^{1'} + x_i^2.x_j^{2'} & 0 & 0 & x_i^1.x_j^2 - x_i^2.x_j^1 \\ 0 & x_i^{1'}.x_j^1 + x_i^2.x_j^{2'} & x_i^{1'}.x_j^2 - x_i^2.x_j^{1'} & 0 \\ 0 & x_i^{2'}.x_j^1 - x_i^1.x_j^{2'} & x_i^1.x_j^{1'} + x_i^{2'}.x_j^2 & 0 \\ x_i^{2'}.x_j^{1'} - x_i^{1'}.x_j^{2'} & 0 & 0 & x_i^{1'}.x_j^1 + x_i^{2'}.x_j^2 \end{bmatrix}$$

The product $X_j.\ X_i$ is obtained in inverting i and j:

(18)

$$(X_j.\ X_i) \text{ if } x_i^0 = 0 \text{ for } i = 1, 2$$

$$=$$

$$\begin{bmatrix} x_j^1.x_i^{1'} + x_j^2.x_i^{2'} & 0 & 0 & x_j^1.x_i^2 - x_j^2.x_i^1 \\ 0 & x_j^{1'}.x_i^1 + x_j^2.x_i^{2'} & x_j^{1'}.x_i^2 - x_j^2.x_i^{1'} & 0 \\ 0 & x_j^{2'}.x_i^1 - x_j^1.x_i^{2'} & x_j^1.x_i^{1'} + x_j^{2'}.x_i^2 & 0 \\ x_j^{2'}.x_i^{1'} - x_j^{1'}.x_i^{2'} & 0 & 0 & x_j^{1'}.x_i^1 + x_j^{2'}.x_i^2 \end{bmatrix}$$

For the following matrix:

(20)

$$\tfrac{1}{2}.\ \{X_1.\ X_2 - X_2.\ X_1\}$$

$$=$$

$$\begin{bmatrix} x_i^1.x_j^{1'} + x_i^2.x_j^{2'} & 0 & 0 & x_i^1.x_j^2 - x_i^2.x_j^1 \\ 0 & x_i^{1'}.x_j^1 + x_i^2.x_j^{2'} & x_i^{1'}.x_j^2 - x_i^2.x_j^{1'} & 0 \\ 0 & x_i^{2'}.x_j^1 - x_i^1.x_j^{2'} & x_i^1.x_j^{1'} + x_i^{2'}.x_j^2 & 0 \\ x_i^{2'}.x_j^{1'} - x_i^{1'}.x_j^{2'} & 0 & 0 & x_i^{1'}.x_j^1 + x_i^{2'}.x_j^2 \end{bmatrix}$$

$$-$$

$$\begin{bmatrix} x_j^1.x_i^{1'} + x_j^2.x_i^{2'} & 0 & 0 & x_j^1.x_i^2 - x_j^2.x_i^1 \\ 0 & x_j^{1'}.x_i^1 + x_j^2.x_i^{2'} & x_j^{1'}.x_i^2 - x_j^2.x_i^{1'} & 0 \\ 0 & x_j^{2'}.x_i^1 - x_j^1.x_i^{2'} & x_j^1.x_i^{1'} + x_j^{2'}.x_i^2 & 0 \\ x_j^{2'}.x_i^{1'} - x_j^{1'}.x_i^{2'} & 0 & 0 & x_j^{1'}.x_i^1 + x_j^{2'}.x_i^2 \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} x_i^1.x_j^{1'} + x_i^2.x_j^{2'} - (x_j^1.x_i^{1'} + x_j^2.x_i^{2'}) & 0 & 0 & x_i^1.x_j^2 - x_i^2.x_j^1 - (x_j^1.x_i^2 - x_j^2.x_i^1) \\ 0 & x_i^{1'}.x_j^1 + x_i^2.x_j^{2'} - (x_j^{1'}.x_i^1 + x_j^2.x_i^{2'}) & x_i^{1'}.x_j^2 - x_i^2.x_j^{1'} - (x_j^{1'}.x_i^2 - x_j^2.x_i^{1'}) & 0 \\ 0 & x_i^{2'}.x_j^1 - x_i^1.x_j^{2'} - (x_j^{2'}.x_i^1 - x_j^1.x_i^{2'}) & x_i^1.x_j^{1'} + x_i^{2'}.x_j^2 - (x_j^1.x_i^{1'} + x_j^{2'}.x_i^2) & 0 \\ x_i^{2'}.x_j^{1'} - x_i^{1'}.x_j^{2'} - (x_j^{2'}.x_i^{1'} - x_j^{1'}.x_i^{2'}) & 0 & 0 & x_i^{1'}.x_j^1 + x_i^{2'}.x_j^2 - (x_j^{1'}.x_i^1 + x_j^{2'}.x_i^2) \end{bmatrix}$$

I should now consider two situations:

1°) either we work on R (or C); it is then easy to check that we don't systematically obtain a diagonal matrix. But this formalism appears to be "a priori" compatible with the one of η (13) if the following ad hoc conditions hold true:

(20-0)

$$\begin{bmatrix} 0 & x_i^{2'}.x_j^1 - x_i^1.x_j^{2'} - (x_j^{2'}.x_i^1 - x_j^1.x_i^{2'}) \\ x_i^{2'}.x_j^{1'} - x_i^{1'}.x_j^{2'} - (x_j^{2'}.x_i^{1'} - x_j^{1'}.x_i^{2'}) & 0 \end{bmatrix} = 0_3$$

(20-1)

$$\begin{bmatrix} 0 & x_i^1.x_j^2 - x_i^2.x_j^1 - (x_j^1.x_i^2 - x_j^2.x_i^1) \\ x_i^{1'}.x_j^2 - x_i^2.x_j^{1'} - (x_j^{1'}.x_i^2 - x_j^2.x_i^{1'}) & 0 \end{bmatrix} = 0_3$$

(20-2)

$$\begin{bmatrix} x_i^1.x_j^{1'} + x_i^2.x_j^{2'} - (x_j^1.x_i^{1'} + x_j^2.x_i^{2'}) & 0 \\ 0 & x_i^{1'}.x_j^1 + x_i^2.x_j^{2'} - (x_j^{1'}.x_i^1 + x_j^2.x_i^{2'}) \end{bmatrix} = -[K]$$

And:
(20-3)

$$\begin{bmatrix} x_i^1.x_j^{1'} + x_i^{2'}.x_j^2 - (x_j^1.x_i^{1'} + x_j^{2'}.x_i^2) & \\ & x_i^{1'}.x_j^1 + x_i^{2'}.x_j^2 - (x_j^{1'}.x_i^1 + x_j^{2'}.x_i^2) \end{bmatrix} = -Id_3$$

2°) or, for the pedagogy, I now work with a set of anti-commutative elements x_i^k and $x_i^{k'}$; this means that:
(21)

$$a.\,b + b.\,a = 0$$

In which case the off-diagonal elements vanish and the matrix (20) is now:
(22)

$$\tfrac{1}{2}.\{X_1.\,X_2 - X_2.\,X_1\}$$

$$=$$

$$\begin{bmatrix} x_i^1.x_j^{1'} + x_i^2.x_j^{2'} - (x_j^1.x_i^{1'} + x_j^2.x_i^{2'}) & & & \\ & x_i^{1'}.x_j^1 + x_i^2.x_j^{2'} - (x_j^{1'}.x_i^1 + x_j^2.x_i^{2'}) & & \\ & & x_i^1.x_j^{1'} + x_i^{2'}.x_j^2 - (x_j^1.x_i^{1'} + x_j^{2'}.x_i^2) & \\ & & & x_i^{1'}.x_j^1 + x_i^{2'}.x_j^2 - (x_j^{1'}.x_i^1 + x_j^{2'}.x_i^2) \end{bmatrix}$$

Despite the appearance, and because of (21), all elements of the diagonal are again equal. That (22) formalism is once more time not compatible with the one of η (13).

Concluding sub-remark

Let examine the only potentially interesting situations corresponding to the relations (20-0, 1, 2 and 3). Because of (20-0 and 1) we have:
(23-0, 1, 2 and 3)

$$x_i^{2'}.x_j^1 - x_i^1.x_j^{2'} = 0$$

$$x_i^{2'}.x_j^{1'} - x_i^{1'}.x_j^{2'} = 0$$

$$x_i^{1'}.x_j^2 - x_i^2.x_j^{1'} = 0$$

$$x_i^1.x_j^2 - x_i^2.x_j^1 = 0$$

Provided none of the element vanishes, these relations imply:
(24)

$$\frac{x_j^{2'}}{x_i^{2'}} = \frac{x_j^1}{x_i^1} = \frac{x_j^{1'}}{x_i^{1'}} = \frac{x_j^2}{x_i^2} = \text{constant}$$

Observing attentively that result (24) allows saying that it is equivalent to the fact that the matrix X_j is proportional to the matrix X_i. In which case, the matrix (20) must vanish, reducing that attempt to nothing.

In that paragraph I have examined the hypothesis consisting to believe that the Minkowskian metric can be represented by a matrix which is itself the representation of a bi-vector “à la E. Cartan [08]”. For this purpose, I have considered all permutations on two indices. There is only one possible permutation; namely (1, 2) → (2, 1). This is explaining the examined formalism (14) which I believe is respecting [08; § 95, p. 83].

Theorem 01

My conclusion is actually a negative one: “There is no matrix representing a 2-vector “à la E. Cartan” which is at the same time a representation of the Minkowskian metric η (13)”.

Complementary sub-remarks 04

If, like me and because of the preliminary argumentation developed previously in starting that section, the reader has a doubt concerning the interpretation of [08; § 95, p. 83], he may examine like us the other alternative (see annex 02) and come to the same negative conclusion.

I nevertheless emit some doubts concerning the previous theorem because of an E. Cartan's remark that has been made in the book [08; § 125, 110, first and second eventualities]. With other words, matrices associated with bi-vectors in a 4D space E_4 have a special structure.

I also want to add some complementary sub-remarks which I believe will help to understand the historic progression surrounding the concept of spinor:
1°) the annex 02 may be seen as a soft prelude to a more general relation that students meet in studying the so-called Dirac's matrices;
2°) the E. Cartan's work [08; p. 44 and p. 112] and more recent works, e. g. [12; chapter 2, §§ 2.1, 2.2 and 2.3, pp. 62-76], give relatively precise indications on the relationship between the concept of spinor and the one of rotation.
3°) even the relation [08; chapter IX, p. 145, (1)] which was the starter for that section introduces some confusion in our spirit. E. Cartan himself is speaking of a bi-vector (at the end of page 145) when referring to a simple product of two matrices A_1 and A_2. This would mean that either (17) or (18) may be representations of bi-vectors; and this would perhaps facilitate our quest in a comparison with (13). This is unfortunately not the case because an attentive observation of the relations (17) and (18) shows that a direct confrontation with (13) yields (24) again. This is implying that only the square of a given matrix can perhaps exhibit the formalism which we are looking for. Unfortunately, the formalism of η is forbidding that eventuality. The best situation that we can hope to obtain with (17) and (18) is a fruitful comparison with the formalism [08; § 125, 110, second eventuality]; and the latter is not directly comparable with the one of η.

At the end of the day, despite all efforts in trying to understand E. Cartan's work, I am obliged to confirm the previous theorem.

Corollary 01

The EM fields of my theory in continuous energetic contexts, (01), are not equivalent to an infinitesimal transformation of the trivial matrix $\blacktriangledown_\Gamma\Phi(^{(4)}\mathbf{p})$ resulting from the action of the Minkowskian metric in the sense given by [08; chapter IX, § 172, (1)]. If I want to push the research into that direction, I must either (a) probably extend the domain of definition yielding (01); a first path is to reconsider the formulation of the Lorentz-Einstein law that I have involved into my approach. Or (b) I should better look for situations where the trivial matrix $\blacktriangledown_\Gamma\Phi^t(^{(4)}\mathbf{p})$ is a bi-vector without being an anti-symmetric matrix. Let remember that the relation [08; chapter IX, § 172, (1)] is concerning simple rotations in a Euclidian geometry. This context may appear to be in contradiction with the wish to analyze a mathematical element involving the Levi-Civita cube which is, of course, the signature for a curved space. *This fact suggests that, even if I success in my quest (= I can find situations for which $\blacktriangledown_\Gamma\Phi^t(^{(4)}\mathbf{p})$ is a bi-vector "à la E. Cartan"), I then shall have to carefully reduce the validity of my conclusion to spaces where the curvature is closed to zero.

Sub-remark 05: exploring the properties of the trivial matrix.

This is the right moment to exploit the calculations of previous paragraphs. Confronting the relation (07) which is explicating the precise formalism of the elements of the trivial matrix $\blacktriangledown_\Gamma\Phi^t(^{(4)}\mathbf{p})$ with the generic formalism of a matrix representing a bi-vector "à la E. Cartan", namely (20), I am left with:

(25)

$$\begin{bmatrix} x_i^1.x_j^{1'} + x_i^2.x_j^{2'} - (x_j^1.x_i^{1'} + x_j^2.x_i^{2'}) & 0 & 0 & 2.(x_i^1.x_j^2 - x_i^2.x_j^1) \\ 0 & x_i^{1'}.x_j^1 + x_i^2.x_j^{2'} - (x_j^{1'}.x_i^1 + x_j^2.x_i^{2'}) & 2.(x_i^{1'}.x_j^2 - x_i^2.x_j^{1'}) & 0 \\ 0 & 2.(x_i^{2'}.x_j^1 - x_i^1.x_j^{2'}) & x_i^1.x_j^{1'} + x_i^{2'}.x_j^2 - (x_j^1.x_i^{1'} + x_j^{2'}.x_i^2) & 0 \\ 2.(x_i^{2'}.x_j^{1'} - x_i^{1'}.x_j^{2'}) & 0 & 0 & x_i^{1'}.x_j^1 + x_i^{2'}.x_j^2 - (x_j^{1'}.x_i^1 + x_j^{2'}.x_i^2) \end{bmatrix}$$

$$=$$

$$\begin{bmatrix} \Gamma^0_{\mu 0}.p^\mu & \Gamma^0_{\mu 1}.p^\mu & \Gamma^0_{\mu 2}.p^\mu & \Gamma^0_{\mu 3}.p^\mu \\ \Gamma^1_{\mu 0}.p^\mu & \Gamma^1_{\mu 1}.p^\mu & \Gamma^1_{\mu 2}.p^\mu & \Gamma^1_{\mu 3}.p^\mu \\ \Gamma^2_{\mu 0}.p^\mu & \Gamma^2_{\mu 1}.p^\mu & \Gamma^2_{\mu 2}.p^\mu & \Gamma^2_{\mu 3}.p^\mu \\ \Gamma^3_{\mu 0}.p^\mu & \Gamma^3_{\mu 1}.p^\mu & \Gamma^3_{\mu 2}.p^\mu & \Gamma^3_{\mu 3}.p^\mu \end{bmatrix}$$

Let observe the formalism attentively and state that the confrontation imposes:

(26-1 to 8)

$$\Gamma_{\mu 1}{}^0.\ p^\mu = \Gamma_{\mu 2}{}^0.\ p^\mu = \Gamma_{\mu 0}{}^1.\ p^\mu = \Gamma_{\mu 3}{}^1.\ p^\mu = \Gamma_{\mu 0}{}^2.\ p^\mu = \Gamma_{\mu 3}{}^2.\ p^\mu = \Gamma_{\mu 1}{}^3.\ p^\mu = \Gamma_{\mu 2}{}^3.\ p^\mu = 0$$

(26-9, 10, 11, 12)

$$a = x_i^1.\ x_j^{1'} + x_i^2.\ x_j^{2'}$$

$$b = x_i^{1'} . x_j^{1} + x_i^{2'} . x_j^{2}$$

$$c = x_i^{1} . x_j^{1'} + x_i^{2'} . x_j^{2}$$

$$d = x_i^{1'} . x_j^{1} + x_i^{2} . x_j^{2'}$$

(26-13 to16)

$$\Gamma^{0}_{\mu 0} . p^{\mu} = a - b = - \Gamma^{1}_{\mu 1} . p^{\mu}$$

$$\Gamma^{2}_{\mu 2} . p^{\mu} = c - d = - \Gamma^{3}_{\mu 3} . p^{\mu}$$

(26-17 to 20)

$$2. (x_i^{2'} . x_j^{1} - x_i^{1} . x_j^{2'}) = \Gamma^{2}_{\mu 1} . p^{\mu}$$

$$2. (x_i^{2'} . x_j^{1'} - x_i^{1'} . x_j^{2'}) = \Gamma^{3}_{\mu 0} . p^{\mu}$$

$$2. (x_i^{1'} . x_j^{2} - x_i^{2} . x_j^{1'}) = \Gamma^{1}_{\mu 2} . p^{\mu}$$

$$2. (x_i^{1} . x_j^{2} - x_i^{2} . x_j^{1}) = \Gamma^{0}_{\mu 3} . p^{\mu}$$

This set of conditions is associated with technical difficulties. It may be decomposed into two subsets of each eight conditions.

The first sub-set coincides with (26-1 to 8) and imposes drastic consequences. Indeed, provided I don't want to restrict a priori the generality on the Levi-Civita cube involved in the discussion (except the facts that that cube is symmetric per definition and that we should stay in the vicinity of a flat space)*, I am forced to accept that the four p^{μ}, for $\mu = 0, 1, 2, 3$, are not independent. This doesn't really surprise me since I know that the generalized theory of relativity (GTR) imposes the relation $p_{\mu} . p^{\mu} = 0$ for massless particles (e.g.: photons, gravitons…) or the dispersion relation for non-vanishing masses. At this stage I yet have (4 x 10 =) 40 different Christoffel's symbols of the second kind and probably only three independent components for the kinetic momentum.

At a first glance, the second subset (26-13 to 20) tells that I have at the end only six independent linear combinations of the $\Gamma_{\mu\beta}^{\nu} . p^{\mu}$ type. And they all potentially depend on the 2 x 4 components of the two vectors that I am looking for. But let examine (26-17 to 20) more attentively:

$$2. (x_i^{2'} . x_j^{1} - x_i^{1} . x_j^{2'}) = \Gamma^{2}_{\mu 1} . p^{\mu}$$

$$2. (x_i^{2'} . x_j^{1'} - x_i^{1'} . x_j^{2'}) = \Gamma^{3}_{\mu 0} . p^{\mu}$$

$$2. (x_i^{1'} . x_j^{2} - x_i^{2} . x_j^{1'}) = \Gamma^{1}_{\mu 2} . p^{\mu}$$

$$2. (x_i^{1} . x_j^{2} - x_i^{2} . x_j^{1}) = \Gamma^{0}_{\mu 3} . p^{\mu}$$

From (26-9 to 12) I infer that:

(27-1 and 2)

$$a . b = (x_i^{1} . x_j^{1'} + x_i^{2} . x_j^{2'}) . (x_i^{1'} . x_j^{1} + x_i^{2'} . x_j^{2}) = x_i^{1} . x_j^{1'} . x_i^{1'} . x_j^{1} + x_i^{1} . x_j^{1'} . x_i^{2'} . x_j^{2} + x_i^{2} . x_j^{2'} . x_i^{1'} . x_j^{1} + x_i^{2} . x_j^{2'} . x_i^{2'} . x_j^{2}$$

$$c . d = (x_i^{1} . x_j^{1'} + x_i^{2'} . x_j^{2}) . (x_i^{1'} . x_j^{1} + x_i^{2} . x_j^{2'}) = x_i^{1} . x_j^{1'} . x_i^{1'} . x_j^{1} + x_i^{1} . x_j^{1'} . x_i^{2} . x_j^{2'} + x_i^{2'} . x_j^{2} . x_i^{1'} . x_j^{1} + x_i^{2'} . x_j^{2} . x_i^{2} . x_j^{2'}$$

Consequently:

(28)

$$a . b - c . d = (x_i^{1} . x_j^{1'} . x_i^{2'} . x_j^{2} + x_i^{2} . x_j^{2'} . x_i^{1'} . x_j^{1}) - (x_i^{1} . x_j^{1'} . x_i^{2} . x_j^{2'} + x_i^{2'} . x_j^{2} . x_i^{1'} . x_j^{1})$$

Simultaneously, I may calculate:

(29)

$$4. (x_i^{1} . x_j^{2} - x_i^{2} . x_j^{1}) . (x_i^{2'} . x_j^{1'} - x_i^{1'} . x_j^{2'}) = 4. (x_i^{1} . x_j^{2} . x_j^{1'} . x_i^{2'} + x_i^{2} . x_j^{2'} . x_i^{1'} . x_j^{1}) - (x_i^{1} . x_j^{2} . x_j^{2'} . x_i^{1'} + x_i^{2} . x_j^{1} . x_j^{1'} . x_i^{2'}) = \Gamma^{0}_{\mu 3} . \Gamma^{3}_{\eta 0} . p^{\mu} . p^{\eta}$$

And:

(30)

$$4. (x_i^{2'} . x_j^{1} - x_i^{1} . x_j^{2'}) . (x_i^{1'} . x_j^{2} - x_i^{2} . x_j^{1'}) = 4. (x_i^{2'} . x_j^{1} . x_j^{2} . x_i^{1'} + x_i^{1} . x_j^{2'} . x_j^{1'} . x_i^{2}) - (x_i^{1} . x_j^{2'} . x_j^{2} . x_i^{1'} + x_i^{2'} . x_j^{1} . x_j^{1'} . x_i^{2}) = \Gamma_{\mu 1}{}^{2} . \Gamma_{\eta 2}{}^{1} . p^{\mu} . p^{\eta}$$

This is amazingly yielding:

(31)

$$(\Gamma_{\mu 3}^{0}.\ \Gamma_{\eta 0}^{3} - \Gamma_{\mu 1}^{2}.\ \Gamma_{\eta 2}^{1}).\ p^{\mu}.\ p^{\eta} = 4.\ (a.\ b - c.\ d)$$

Note at this stage that, except perhaps for (26-13 to 16), all other arrangements of the type (31) -where only the lateral indices of the Christoffel's symbols are changing- vanish because of (26-1 to 8). Note also that the relation (31) is totally independent on the components of the two vectors I am supposed to look for. This is suggesting some algebraic indications; for example, I may up to now affirm that if $\blacktriangledown_{\Gamma}\Phi^{t}({}^{(4)}\mathbf{p})$ represents a bi-vector, then $\blacktriangledown_{\Gamma}\Phi$ cannot be, in general, a multiplicative morphism and the extended product built upon the Levi-Civita cube is not, in general, an associative inner product.

Bivector "à la E. Cartan" and trivial matrix

Proposition 02 : the trivial matrix is a bivector "à la E. Cartan"

Coming back to the initial interrogation, I can say that any matrix with the formalism (17) where a, b, c and d are given via (26-9 to 12) can be written:

(32)

$$\begin{bmatrix} a & 0 & 0 & (x_i^1.x_j^2 - x_i^2.x_j^1) \\ 0 & b & (x_i^{1'}.x_j^2 - x_i^2.x_j^{1'}) & 0 \\ 0 & (x_i^{2'}.x_j^1 - x_i^1.x_j^{2'}) & c & 0 \\ (x_i^{2'}.x_j^{1'} - x_i^{1'}.x_j^{2'}) & 0 & 0 & d \end{bmatrix}$$

The proposition writes now: It is plausibly and simultaneously: (a) the trivial matrix $\blacktriangledown_{\Gamma}\Phi^{t}({}^{(4)}\mathbf{p})$ with a non-vanishing Christoffel's cube and (b) the representation of a bi-vector "à la E. Cartan" each time that:

A. either [08; § 125, p. 110, first eventuality]: 0 = $(x_i^1.x_j^2 - x_i^2.x_j^1)$ = $(x_i^{1'}.x_j^2 - x_i^2.x_j^{1'})$ and $(x_i^{2'}.x_j^1 - x_i^1.x_j^{2'})$ = $(x_i^{2'}.x_j^{1'} - x_i^{1'}.x_j^{2'})$ = 0 if a, b, c, d are correctly co-related;

B. or [08; § 125, p. 110, second eventuality]: 0 = a = b and 0 = c = d if the other terms have the ad hoc formalism.

Characterizations – eventuality A

Let examine the first eventuality [08; § 125, p. 110, first eventuality]. I propose in fact:

(33)

$$\begin{bmatrix} a & 0 & 0 & (x_i^1.x_j^2 - x_i^2.x_j^1) \\ 0 & b & (x_i^{1'}.x_j^2 - x_i^2.x_j^{1'}) & 0 \\ 0 & (x_i^{2'}.x_j^1 - x_i^1.x_j^{2'}) & c & 0 \\ (x_i^{2'}.x_j^{1'} - x_i^{1'}.x_j^{2'}) & 0 & 0 & d \end{bmatrix}$$

=

$$\begin{bmatrix} \Gamma_{\mu 0}^{0}.\ p^{\mu} & \Gamma_{\mu 1}^{0}.\ p^{\mu} & \Gamma_{\mu 2}^{0}.\ p^{\mu} & \Gamma_{\mu 3}^{0}.\ p^{\mu} \\ \Gamma_{\mu 0}^{1}.\ p^{\mu} & \Gamma_{\mu 1}^{1}.\ p^{\mu} & \Gamma_{\mu 2}^{1}.\ p^{\mu} & \Gamma_{\mu 3}^{1}.\ p^{\mu} \\ \Gamma_{\mu 0}^{2}.\ p^{\mu} & \Gamma_{\mu 1}^{2}.\ p^{\mu} & \Gamma_{\mu 2}^{2}.\ p^{\mu} & \Gamma_{\mu 3}^{2}.\ p^{\mu} \\ \Gamma_{\mu 0}^{3}.\ p^{\mu} & \Gamma_{\mu 1}^{3}.\ p^{\mu} & \Gamma_{\mu 2}^{3}.\ p^{\mu} & \Gamma_{\mu 3}^{3}.\ p^{\mu} \end{bmatrix}$$

It is presumably yielding:

$$\frac{x_j^1}{x_i^1} = \frac{x_j^{1'}}{x_i^{1'}} = \frac{x_j^2}{x_i^2} = \text{constant n°1 and } \frac{x_j^{2'}}{x_i^{2'}} = \frac{x_j^1}{x_i^1} = \frac{x_j^{1'}}{x_i^{1'}} = \text{constant n°2}$$

This is of course imposing the equality of the two constants and, consequently, the relation (24) again. The two vectors are proportional and injecting this fact in (17) leaves us with a result of the E. Cartan's theory which is known under the name of "fundamental theorem" [08; § 94, pp. 82-83]. Namely: if the unique remaining vector, ${}^{(4)}\mathbf{x}$, has now the components (x^1, x^2, $x^{1'}$, $x^{2'}$), the Cartan's matrix representation of its square only is the $<{}^{(4)}\mathbf{x},\ {}^{(4)}\mathbf{x}>_{Id}.\ Id_4$ matrix where the $<\dots,\ \dots>_{Id}$ denotes the Euclidian scalar product in E_4. Confronting this with the proposition (33) induces:

(34)

$$\Gamma_{\mu 0}^{0}.\ p^{\mu} = \Gamma_{\mu 1}^{1}.\ p^{\mu} = \Gamma_{\mu 2}^{2}.\ p^{\mu} = \Gamma_{\mu 3}^{3}.\ p^{\mu} = <{}^{(4)}\mathbf{x},\ {}^{(4)}\mathbf{x}>_{Id}$$

Taking a classical result of the GTR into consideration [13; § 35, 101], this is nothing but:

(35)

$$\frac{\partial_{\mu}\ |g|^{1/2}}{|g|^{1/2}}.\ p^{\mu} = <{}^{(4)}\mathbf{x},\ {}^{(4)}\mathbf{x}>_{Id}$$

Following the same approach than the one made with the relation (09), this is also:

(36)

m. $\frac{d|g|^{1/2}}{ds}$ = $|g|^{1/2}$. <$^{(2)}$**x**, $^{(2)}$**x**>$_{Id}$ or equivalently (m ≠ 0): $\frac{d|g|^{1/2}}{|g|^{1/2}} = \frac{1}{m}$. <$^{(4)}$**x**, $^{(4)}$**x**>$_{Id}$. ds

Although the mental gymnastics is interesting, I have no idea about the meaning of the vector $^{(4)}$**x**. And, in some way, I may say that the unicity of that vector is changing the nature of the quest.

Characterizations – eventuality B

The identifications

Let now examine the other possibility [08; § 125, 110, second eventuality]. I immediately conclude that it corresponds to the case where a = b = c = d = 0, thus to a null diagonal and, consequently, to a special case of the previous paragraph; precisely, this case is characterized by:
(37)

$$|g|^{1/2} = \text{invariant}$$

The determinant of the metric tensor (but *not the metric itself*) is invariant; a condition to which I must add:
(26-1 to 8)

$$\Gamma_{\mu1}{}^{0}.\ p^{\mu} = \Gamma_{\mu2}{}^{0}.\ p^{\mu} = \Gamma_{\mu0}{}^{1}.\ p^{\mu} = \Gamma_{\mu3}{}^{1}.\ p^{\mu} = \Gamma_{\mu0}{}^{2}.\ p^{\mu} = \Gamma_{\mu3}{}^{2}.\ p^{\mu} = \Gamma_{\mu1}{}^{3}.\ p^{\mu} = \Gamma_{\mu2}{}^{3}.\ p^{\mu} = 0$$

And I must also add:
(38-1 to 4)

$$(x_i^{2'}.x_j^{1} - x_i^{1}.x_j^{2'}) = \Gamma^{2}_{\mu1}.\, p^{\mu}$$

$$(x_i^{2'}.x_j^{1'} - x_i^{1'}.x_j^{2'}) = \Gamma^{3}_{\mu0}.\, p^{\mu}$$

$$(x_i^{1'}.x_j^{2} - x_i^{2}.x_j^{1'}) = \Gamma^{1}_{\mu2}.\, p^{\mu}$$

$$(x_i^{1}.x_j^{2} - x_i^{2}.x_j^{1}) = \Gamma^{0}_{\mu3}.\, p^{\mu}$$

Remark

The relations (26-1 to 8) are all compatible with the symmetry of $\blacktriangledown_\Gamma \Phi^t(^{(4)}\mathbf{p})$ – see the sub remark (a necessary condition). Concerning the relations (38-1 to 4) I can make a remark: if one adds that necessary condition, these four relations are immediately reduced to only two because that condition imposes:

$$\Gamma_{\mu1}{}^{2} = \Gamma_{\mu2}{}^{1} \text{ and } \Gamma_{\mu0}{}^{3} = \Gamma_{\mu3}{}^{0}$$

Consequently one inherits of:
(39)

$$\forall\ ^{(4)}\mathbf{p}$$

$$(x_i^{1}.x_j^{2} - x_i^{2}.x_j^{1}) = \Gamma^{0}_{\mu3}.\, p^{\mu} = \aleph = \Gamma^{3}_{\mu0}.\, p^{\mu} = (x_i^{2'}.x_j^{1'} - x_i^{1'}.x_j^{2'})$$

$$(x_i^{2'}.x_j^{1} - x_i^{1}.x_j^{2'}) = \Gamma^{2}_{\mu1}.\, p^{\mu} = \Upsilon = \Gamma^{1}_{\mu2}.\, p^{\mu} = (x_i^{1'}.x_j^{2} - x_i^{2}.x_j^{1'})$$

I stay now with only two independent entities: Υ and $\aleph$.

Conditions of coherence: "How many independent components for the bi-vector?"

I have to verify if that new context doesn't generate some incoherence between the diverse products $x_i^a.\ x_j^b$ because of the relations (26-9 to 12). They imply here:
(40-1, 2, 3, 4)

$$a = x_i^{1}.\ x_j^{1'} + x_i^{2}.\ x_j^{2'} = 0$$

$$b = x_i^{1'}.\ x_j^{1} + x_i^{2'}.\ x_j^{2} = 0$$

$$c = x_i^{1}.\ x_j^{1'} + x_i^{2'}.\ x_j^{2} = 0$$

$$d = x_i^{1'}.\ x_j^{1} + x_i^{2}.\ x_j^{2'} = 0$$

Or more precisely:

$$\frac{x_i^2}{x_i^1} = -\frac{x_j^{1'}}{x_j^{2'}}\ ;\ \frac{x_j^2}{x_j^1} = -\frac{x_i^{1'}}{x_i^{2'}}\ ;\ \frac{x_j^2}{x_i^1} = -\frac{x_j^{1'}}{x_i^{2'}}\ ;\ \frac{x_j^1}{x_i^2} = -\frac{x_j^{2'}}{x_i^{1'}}\ ;$$

Initially, the purpose is to manipulate two any vectors ${}^{(4)}\mathbf{x}_i$ and ${}^{(4)}\mathbf{x}_j$ in minimizing the reduction of the generality of the Levi-Civita cube. If these two vectors are really any one, then the r.h.t. of the relations (38-1 to 4) are also any one. But one now suspect that the relations (40-1 to 4) introduce some restrictions; in extenso: one has no more 8 independent components. The next question is then: "How many independent components do I have?" Let suppose that ${}^{(4)}\mathbf{x}_i$ has the known and non-vanishing components (x_i^1, x_i^2, $x_i^{1'}$, $x_i^{2'}$). The second vector is ${}^{(4)}\mathbf{x}_j$ and has the unknown components (x_j^1, x_j^2, $x_j^{1'}$, $x_j^{2'}$). As a first indication, because of (40-3) one gets:

$${}^{(4)}\mathbf{x}_j: (x_j^1,\ -\frac{x_j^{1'}}{x_i^{2'}}.\ x_i^1,\ x_j^{1'},\ x_j^{2'})$$

And because of (40-1) I finally get:
(41)

$${}^{(4)}\mathbf{x}_j: (x_j^1,\ -\frac{x_j^{1'}}{x_i^{2'}}.\ x_i^1,\ x_j^{1'},\ -\ x_i^1.\frac{x_j^{1'}}{x_i^2})$$

This is suggesting that the second and unknown vector which I am looking for has at most two independent components; namely: x_j^1 and $x_j^{1'}$. But let observe (40-1 to 4) more attentively and state that the choice of (40-1) and (40-3) which has been made to reduce the number of independent components was arbitrary. Because of that, my conclusion is still incomplete. Of course, with (40-1) and (40-4) I have two different possibilities to rewrite $x_j^{2'}$:

$$x_j^{2'} = -\ x_i^1.\frac{x_j^{1'}}{x_i^2} = -\ x_i^{1'}.\frac{x_j^1}{x_i^2}$$

This is yielding:
(42)

$$x_i^1.\ x_j^{1'} - x_i^{1'}.\ x_j^1 = 0$$

On the same vein, considering (40-2) and (40-3), I state that I have two different possibilities to write x_j^2:

$$x_j^2 = -\ x_j^1.\frac{x_i^{1'}}{x_i^{2'}} = -\ x_i^1.\frac{x_j^{1'}}{x_i^{2'}}$$

Once again, this is yielding:

$$x_i^1.\ x_j^{1'} - x_i^{1'}.\ x_j^1 = 0$$

Confronting now (40-1) and (40-3) I may state the existence of two possibilities for $x_j^{1'}$:

$$x_j^{1'} = -\ x_j^{2'}.\frac{x_i^2}{x_i^1} = -\ x_i^{2'}.\frac{x_j^2}{x_i^1}$$

This is yielding:
(43)

$$x_i^2.\ x_j^{2'} - x_i^{2'}.\ x_j^2 = 0$$

These diverse relations are completely changing the understanding of the relations (38-1 to 4). Indeed, let inject the results contained in (41) into the relations (38-1 to 4). They now write:
(44-1 to 4)

$$x_i^{2'}.\ x_j^1 + (x_i^1)^2.\frac{x_j^{1'}}{x_i^2} = \Gamma^2_{\mu 1}.\ p^\mu$$

$$x_i^{2'}.\ x_j^{1'} + x_i^{1'}.\ x_i^1.\frac{x_j^{1'}}{x_i^2} = \Gamma^3_{\mu 0}.\ p^\mu = x_j^{1'}.\ (x_i^{2'} + x_i^1.\frac{x_i^{1'}}{x_i^2})$$

$$(x_i^{1'}.-\frac{x_j^{1'}}{x_i^{2'}}.x_i^1 - x_i^2.x_j^{1'}) = \Gamma^1_{\mu 2}.\ p^\mu = -\ x_j^{1'}.\ (\frac{x_i^{1'}}{x_i^{2'}}.x_i^1 + x_i^2)$$

$$(x_i^1.x_j^2 - x_i^2.x_j^1) = \Gamma^0_{\mu 3}.\ p^\mu$$

But this is not all. A difficulty is to get a panoramic view on all possible products. Let write:
(45)

$$T(\otimes)({}^{(4)}\mathbf{x}_i,\ {}^{(4)}\mathbf{x}_j)$$

=

x_j^1	$x_i^1 . x_j^1$	$x_i^2 . x_j^1$	$x_i^{1'} . x_j^1$	$x_i^{2'} . x_j^1$
x_j^2	$x_i^1 . x_j^2$	$x_i^2 . x_j^2$	$x_i^{1'} . x_j^2$	$x_i^{2'} . x_j^2$
$x_j^{1'}$	$x_i^1 . x_j^{1'}$	$x_i^2 . x_j^{1'}$	$x_i^{1'} . x_j^{1'}$	$x_i^{2'} . x_j^{1'}$
$x_j^{2'}$	$x_i^1 . x_j^{2'}$	$x_i^2 . x_j^{2'}$	$x_i^{1'} . x_j^{2'}$	$x_i^{2'} . x_j^{2'}$
$\otimes$	x_i^1	x_i^2	$x_i^{1'}$	$x_i^{2'}$

A special case is the transposed of that matrix:
(46)

$$T(\otimes)({}^{(4)}\mathbf{x}_j, {}^{(4)}\mathbf{x}_i)$$

$$=$$

$$\begin{bmatrix} x_j^1 . x_i^1 & x_j^2 . x_i^1 & x_j^{1'} . x_i^1 & x_j^{2'} . x_i^1 \\ x_j^1 . x_i^2 & x_j^2 . x_i^2 & x_j^{1'} . x_j^2 & x_j^{2'} . x_i^2 \\ x_j^1 . x_i^{1'} & x_j^2 . x_i^{1'} & x_j^{1'} . x_i^{1'} & x_j^{2'} . x_i^{1'} \\ x_j^1 . x_i^{2'} & x_j^2 . x_i^{2'} & x_j^{1'} . x_i^{2'} & x_j^{2'} . x_i^{2'} \end{bmatrix}$$

Let try to simplify the matrix (45). Because of (42) and (43):
(47)

$$T(\otimes)({}^{(4)}\mathbf{x}_i, {}^{(4)}\mathbf{x}_j)$$

$$=$$

$$\begin{bmatrix} x_i^1 . x_j^1 & x_i^2 . x_j^1 & x_i^1 . x_j^{1'} & x_i^{2'} . x_j^1 \\ x_i^1 . x_j^2 & x_i^2 . x_j^2 & x_i^{1'} . x_j^2 & x_i^2 . x_j^{2'} \\ x_i^1 . x_j^{1'} & x_i^2 . x_j^{1'} & x_i^{1'} . x_j^{1'} & x_i^{2'} . x_j^{1'} \\ x_i^1 . x_j^{2'} & x_i^2 . x_j^{2'} & x_i^{1'} . x_j^{2'} & x_i^{2'} . x_j^{2'} \end{bmatrix}$$

And the four relations (40-1 to 4) are in fact only one:
(48)

$$a = x_i^1 . x_j^{1'} + x_i^2 . x_j^{2'} = 0$$

$$b = x_i^{1'} . x_j^1 + x_i^{2'} . x_j^2 = 0$$

$$c = x_i^1 . x_j^{1'} + x_i^{2'} . x_j^2 = 0$$

$$d = x_i^{1'} . x_j^1 + x_i^2 . x_j^{2'} = 0$$

This is implying a new simplification:
(49)

$$T(\otimes)({}^{(4)}\mathbf{x}_i, {}^{(4)}\mathbf{x}_j)$$

$$=$$

$$\begin{bmatrix} x_i^1 . x_j^1 & x_i^2 . x_j^1 & x_i^1 . x_j^{1'} & x_i^{2'} . x_j^1 \\ x_i^1 . x_j^2 & x_i^2 . x_j^2 & x_i^{1'} . x_j^2 & -x_i^1 . x_j^{1'} \\ x_i^1 . x_j^{1'} & x_i^2 . x_j^{1'} & x_i^{1'} . x_j^{1'} & x_i^{2'} . x_j^{1'} \\ x_i^1 . x_j^{2'} & -x_i^1 . x_j^{1'} & x_i^{1'} . x_j^{2'} & x_i^{2'} . x_j^{2'} \end{bmatrix}$$

Now let observe (44-1 to 4) again and especially (44-2 and 3):
(50-1)

$$x_i^{2'} . x_j^1 + (x_i^1)^2 . \frac{x_j^{1'}}{x_i^2} = \Gamma^2_{\mu 1} . p^\mu \Rightarrow x_i^2 . \Gamma^2_{\mu 1} . p^\mu = x_i^2 . x_i^{2'} . x_j^1 + (x_i^1)^2 . x_j^{1'}$$

(50-2)

$$\Gamma^3_{\mu 0} . p^\mu = x_j^{1'} . (x_i^{2'} + x_i^1 . \frac{x_i^{1'}}{x_i^2}) \Rightarrow x_i^2 . \Gamma^3_{\mu 0} . p^\mu = x_j^{1'} . (x_i^1 . x_i^{1'} + x_i^2 . x_i^{2'})$$

(50-3)

$$\Gamma^1_{\mu 2} . p^\mu = - x_j^{1'} . (\frac{x_i^{1'}}{x_i^{2'}} . x_i^1 + x_i^2) \Rightarrow x_i^{2'} . \Gamma^1_{\mu 2} . p^\mu = - x_j^{1'} . (x_i^1 . x_i^{1'} + x_i^2 . x_i^{2'})$$

Let now eliminate x_j^2 in (44-4) with the help of (41):

(50-4)

$$-\left(x_i^1.\frac{x_j^{1'}}{x_i^{2'}}.x_i^1 + x_i^2.x_j^1\right) = \Gamma^0_{\mu 3}.p^\mu \Rightarrow x_i^{2'}.\Gamma^0_{\mu 3}.p^\mu = -(x_i^2.x_i^{2'}.x_j^1 + (x_i^1)^2.x_j^{1'})$$

The situation is quite more complicated than initially expected. Indeed, because of Cartan's theory and because of the use I want to make of it, I started with a priori 8 components (four for each of the two 4D-vectors I am looking for) connected via 4 relations (38-1 to 4).

These relations could have been in fact understood as defining 6 planes (each pair of these 4 relations defines a plane and one can choice 6 different pairs). These planes are obviously not independent. Note that four planes in a 3D space are sufficient to build, e.g. a tetrahedral pyramid. The two supplementary planes of that theory would have told something about the time. Note also that two bi-vectors are a sufficient tool to build a 3D tetrahedral pyramid.

Now, because of the consequences of (40-1 to 4), in particular the relations (42), (43), (48) and (50-1 to 4), I must state that supposing (a) that (33) hold true in case **B.** ([08; § 125, 110, second eventuality] the main hypothesis here) and (b) that $^{(4)}\mathbf{x}_i$ has the known and non-vanishing components (x_i^1, x_i^2, $x_i^{1'}$, $x_i^{2'}$) are sufficient prerequisites to discover the missing half of a second vector $^{(4)}\mathbf{x}_j$ of which two components would be known (x_j^1, …, $x_j^{1'}$, …).
There are only 6 independent components in the two unknown vectors. Furthermore, these 6 components impose strict conditions to the remaining non-vanishing terms of the trivial matrix $\blacktriangledown_\Gamma\Phi^t(^{(4)}\mathbf{p})$.

In writing (33) for the configuration **B.** [08; § 125, 110, second eventuality], I had the illusion to dispose of four independent eventually non-vanishing terms in that trivial matrix. The relations (50-1 to 4) are clearly indicating that at most two of them are independent and eventually non-vanishing terms.
(51-1 and 2)

$$x_i^2.\Gamma^2_{\mu 1}.p^\mu = x_i^2.x_i^{2'}.x_j^1 + (x_i^1)^2.x_j^{1'} = -x_i^{2'}.\Gamma^0_{\mu 3}.p^\mu$$

$$x_i^2.\Gamma^3_{\mu 0}.p^\mu = x_j^{1'}.(x_i^1.x_i^{1'} + x_i^2.x_i^{2'}) = -x_i^{2'}.\Gamma^1_{\mu 2}.p^\mu$$

Confrontation with the necessary conditions of symmetry

Note that the relations (51) have been obtained without taking attention to the consequence of the necessary conditions of symmetry (39). Let now inject them in the discussion. Whatever the components of the two 4D vectors:

$$x_i^2.\Upsilon = -x_i^{2'}.\aleph = x_i^2.x_i^{2'}.x_j^1 + (x_i^1)^2.x_j^{1'}$$

$$x_i^2.\aleph = -x_i^{2'}.\Upsilon = x_j^{1'}.(x_i^1.x_i^{1'} + x_i^2.x_i^{2'})$$

Let multiply the first identity by $x_i^{2'}$:
(51-3)

$$x_i^{2'}.x_i^2.\Upsilon = -(x_i^{2'})^2.\aleph$$

$$\Downarrow$$

$$(-x_i^{2'}.\Upsilon).(-x_i^2) = x_i^2.\aleph.(-x_i^2) = -(x_i^{2'})^2.\aleph$$

$$\Downarrow$$

$$x_i^2 = \pm x_i^{2'}$$

Confronting this with (43) imposes:
(51-4)

$$x_j^2 = \pm x_j^{2'}$$

In multiplying the second identity by x_i^2 I obtain exactly the same constraint. The first 4D vector is consequently reduced to the following formalisms:
(51-5)

$$^{(4)}\mathbf{x}_i: (x_i^1, x_i^2, x_i^{1'}, \pm x_i^2)$$

And, because of (41), the second vector is reduced to the following formalisms:
(51-6)

$$^{(4)}\mathbf{x}_j: \left(x_j^1, -\frac{x_j^{1'}}{x_i^2}.x_i^1, x_j^{1'}, \pm x_i^1.\frac{x_j^{1'}}{x_i^2}\right)$$

Both 4D vectors seems to be very similar in the sense that (a) three scalars are enough to define them and (b) these scalars are positioned in a similar manner. Nevertheless, looking more attentively to the details is showing that the second vector

is entirely defined by the first one and two new scalars only; namely: x_j^1 and $x_j^{1'}$. At the end, injecting (51-3) into (51-1 or 2) yields:
(51-7)

$$\Upsilon = \pm \aleph$$

The Pythagorean table of (tensor) multiplication (49) can be simplified a step further:
(51-8)

$$T(\otimes)({}^{(4)}\mathbf{x}_i, {}^{(4)}\mathbf{x}_j) =$$

$$\begin{bmatrix} x_i^1.\, x_j^1 & x_i^2.\, x_j^1 & x_i^1.\, x_j^{1'} & \pm x_i^2.\, x_j^1 \\ x_i^1.\, x_j^2 & x_i^2.\, x_j^2 & x_i^{1'}.\, x_j^2 & -x_i^1.\, x_j^{1'} \\ x_i^1.\, x_j^{1'} & x_i^2.\, x_j^{1'} & x_i^{1'}.\, x_j^{1'} & \pm x_i^2.\, x_j^{1'} \\ \pm x_i^1.\, x_j^2 & -x_i^1.\, x_j^{1'} & \pm x_i^{1'}.\, x_j^2 & \pm x_i^2.\, x_j^2 \end{bmatrix}$$

Mathematical conclusion

The relation (51-7) introduces one more restriction and one is left with only one eventually non-systematically vanishing linear combination of the Christoffel's symbols. This is imposing a very specific formalism for the trivial matrix of the discussion:

$$\blacktriangledown_\Gamma \Phi^t({}^{(4)}\mathbf{p}) = \begin{bmatrix} 0 & 0 & 0 & \aleph \\ 0 & 0 & \Upsilon & 0 \\ 0 & \Upsilon & 0 & 0 \\ \aleph & 0 & 0 & 0 \end{bmatrix} \text{ and } \Upsilon = \pm \aleph$$

Concretely this will be one of the two possible configurations:

$$\blacktriangledown_\Gamma \Phi^t({}^{(4)}\mathbf{p}) = \begin{bmatrix} 0 & 0 & 0 & \aleph \\ 0 & 0 & \aleph & 0 \\ 0 & \aleph & 0 & 0 \\ \aleph & 0 & 0 & 0 \end{bmatrix} \text{ or } \blacktriangledown_\Gamma \Phi^t({}^{(4)}\mathbf{p}) = \begin{bmatrix} 0 & 0 & 0 & \aleph \\ 0 & 0 & -\aleph & 0 \\ 0 & -\aleph & 0 & 0 \\ \aleph & 0 & 0 & 0 \end{bmatrix}$$

For convenience, I propose to write:

$$[0] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ [\Xi] = \begin{bmatrix} 0 & \chi \\ \chi & 0 \end{bmatrix} \text{ and } [H] = \begin{bmatrix} 0 & -\chi \\ \chi & 0 \end{bmatrix} \text{ with } \chi = \Gamma_{\mu 3}{}^0.\, p^\mu$$

With that notation, the allowed trivial matrices for the investigation at hand are:

$$\blacktriangledown_\Gamma \Phi^t({}^{(4)}\mathbf{p}) = \begin{bmatrix} [0] & [\Xi] \\ [\Xi] & [0] \end{bmatrix} \text{ or } \blacktriangledown_\Gamma \Phi^t({}^{(4)}\mathbf{p}) = \begin{bmatrix} [0] & -[H] \\ [H] & [0] \end{bmatrix}$$

I am presently working in an $E_4 = E_{2\times 2}$ space; thus $\nu = 2$. Each of the matrices [Ξ] and [H] introduced here is of degree $2^{\nu - 1} = 2^{2-1} = 2$ as expected (see footnote [08; p. 110]. Both configurations seem luckily to be compatible with the one suggested by E. Cartan [08; p. 110] for matrices associated with 2-vectors (just another name for the bi-vectors).

Considering an example of constraints coming from physics

Are all sides of the technical difficulties inspected? No. One must also not forget another preliminary remark: that exploration holds true in contexts with a very small curvature. This is the reason why that vision must be more concretely tested, for example with the help of the linearized version of the A. Einstein's theory; see in [16; §4, p.7, (16)]. Following the notation of that document [16], we get:
(52-1 and 2)

$$\Gamma^3_{\mu 0}.\, p^\mu = (\partial_3\phi + \partial_t\omega_3).\, p^0 + (\partial_{[k}\omega_{3]} + \partial_t s_{k3}).\, p^k - \Psi.\, p^3$$

$$\Gamma^0_{\mu 3}.\, p^\mu = \partial_3\phi.\, p^0 + (\partial_t s_{k3} - \partial_{(k}\omega_{3)}).\, p^k - \Psi.\, p^3$$

From which e deduce:
(53)

$$½.\, (\Gamma^3_{\mu 0} - \Gamma^0_{\mu 3}).\, p^\mu = \partial_k\omega_3.\, p^k$$

Since the trivial matrix must be symmetric, this is imposing the first concrete constraint following from the hypothesis which has been made in that document; namely:
(54)

$$\partial_k\omega_3.\, p^k = 0$$

This is in fact the first limpid conclusion resulting from that complicated and exhaustive confrontation. Indeed, looking at the meaning of the ω_3 term in the reference document [16; §3, p. 4, (11)] we discover that it represents one of the term of the 4D metric, namely h_{03}. An expression like (54) is suggesting that that term is spatially preserved. That term plays an interesting role in considerations centered on the Thirring-Lense effect because it is directly related to a gravito-magnetism mechanism. We also have:

(55-1)

$$\Gamma^2_{\mu 1}.\ p^\mu = (\partial_{[1}\omega_{2]} + \partial_t s_{12} - \delta_{12}.\ \Psi).\ p^0 + (\delta_{k1}.\ \partial_2\Psi - 2.\ \delta_{(k}\partial_{1)}\Psi - \partial_2 s_{k1} + 2.\ \partial_{(k}s_{1)2}).\ p^k$$

(55-2)

...

Because of (51-1), we get:

(56)

$$x_i{}^2.\ \{(\partial_{[1}\omega_{2]} + \partial_t s_{12} - \delta_{12}.\ \Psi).\ p^0 + (\delta_{k1}.\ \partial_2\Psi - 2.\ \delta_{(k}\partial_{1)}\Psi - \partial_2 s_{k1} + 2.\ \partial_{(k}s_{1)2}).\ p^k\} = x_i{}^2.\ x_i{}^{2'}.\ x_j{}^1 + (x_i{}^1)^2.\ x_j{}^{1'} = -\ x_i{}^{2'}.\ \{\partial_3\phi.\ p^0 + (\partial_t s_{k3} - \partial_{(k}\omega_{3)} - \delta_{k3}.\ \Psi).\ p^k\}$$

The understanding of these equations is far from evident. Although it doesn't represent a complete treatment, we may eventually propose identifications not depending on the components of the kinetic momentum:

$$x_i{}^2.\ (\partial_{[1}\omega_{2]} + \partial_t s_{12} - \delta_{12}.\ \Psi) = -\ x_i{}^{2'}.\ \partial_3\phi$$

$$\forall\ k = 1, 2, 3:\ x_i{}^2.\ (\delta_{k1}.\ \partial_2\Psi - 2.\ \delta_{(k}\partial_{1)}\Psi - \partial_2 s_{k1} + 2.\ \partial_{(k}s_{1)2}) = -\ x_i{}^{2'}.\ (\partial_t s_{k3} - \partial_{(k}\omega_{3)} - \delta_{k3}.\ \Psi)$$

Because of (51-3) this is reduced to:

$$\partial_{[1}\omega_{2]} + \partial_t s_{12} - \delta_{12}.\ \Psi = \pm\ \partial_3\phi$$

$$\forall\ k = 1, 2, 3:\ \delta_{k1}.\ \partial_2\Psi - 2.\ \delta_{(k}\partial_{1)}\Psi - \partial_2 s_{k1} + 2.\ \partial_{(k}s_{1)2}) = \pm\partial_t s_{k3} - \partial_{(k}\omega_{3)} - \delta_{k3}.\ \Psi$$

As mathematician we have no idea about what to do with this. We just state that the mathematical hypothesis imposes precise relations between data coming from physics in the context we want to explore.

Additional constraints

It has already been seen, with the relations (27-1, 2) to (30), that:

$$(\Gamma_{\mu 3}{}^0.\ \Gamma_{\eta 0}{}^3 - \Gamma_{\mu 1}{}^2.\ \Gamma_{\eta 2}{}^1).\ p^\mu.\ p^\eta = (a.\ b - c.\ d)$$

In the specific configuration here, this is evidently reduced to:

$$(\Gamma_{\mu 3}{}^0.\ \Gamma_{\eta 0}{}^3 - \Gamma_{\mu 1}{}^2.\ \Gamma_{\eta 2}{}^1).\ p^\mu.\ p^\eta = 0$$

Without doing any supplementary calculations we must remark that any product of two of the 4 scalars a, b, c or d minus the product of the two remaining scalars vanishes. Because of the specific context which is presently studied: a = b = c = d = 0. This is suggesting that that context is compatible with a simultaneous realization of:

- ✓ $\blacktriangledown_\Gamma\Phi^t({}^{(4)}\mathbf{p})$ as representing a bi-vector "à la E. Cartan";
- ✓ $\blacktriangledown_\Gamma\Phi$ being a multiplicative morphism connecting $E_4(R)$ and $M_4(R)$ via a deformed tensor product built on the Levi-Civita cube; note also that if $\blacktriangledown_\Gamma\Phi$ is a multiplicative morphism then the components of the curvature tensor reduce to the partial derivatives of the Christoffel's symbols of the second kind.
- ✓ The deformed tensor product built upon the Levi-Civita cube as being an associative inner product.

Theorem 02

In that part, I have explored situations realizing the ideas sketched by E. Cartan in [08; § 125, p. 110, second eventuality] and came to the provisory conclusion (because all the additional constraints have not yet been explored in extenso) that, yes, the new formalism of the EM tensor, (01), obtained with the theory of the (E) question in [a] for continuous energetic contexts can be interpreted as an infinitesimal transformation of the 4D metric resulting from the action of the trivial matrix $\blacktriangledown_\Gamma\Phi({}^{(4)}\mathbf{p})$ when the latter is a "bi-vector à la E. Cartan".

That eventuality is mathematically possible but associated with very strong and strict constraints exposed previously in the text. The constraints concern the components of the two vectors constituting the bi-vector and the Christoffel's symbols of the second kind.

At the end of the day, *<u>for symmetric metrics only</u>*, it seems to be a realistic ambition to propose the existence of circumstances such that:
(01-Periat)

$$q. [F_{\mu\nu}] = ½. \{^{(4)}G. \,_{\blacktriangledown\Gamma}\Phi(^{(4)}\mathbf{p}) - \,_{\blacktriangledown\Gamma}\Phi^t(^{(4)}\mathbf{p}). \,^{(4)}G\} = \delta^{(4)}G$$

Although that expression looks formally "simple", it tells the important information for physics; namely: some EM fields are equivalent to infinitesimal variations of the metric in a 4D context. I ignore if the special and limited interpretation of the new formalism, (01-Periat), is really related to some piezoelectrical phenomenon. That interpretation needs further investigations and tests. I shall only note that another investigation, [c], yields a similar suggestion.

Corollary 02: premonition for the existence of a plausible operation on bi-vectors

Since any (2, 0) representation of the EM tensor is a bi-vector [18; p. 560, at the beginning of § 4], the relation (01-Periat) is suggesting the formal existence of some mathematical operation characterized by the action of the representation of an Hermitian spinor, the customized metric tensor $^{(4)}G$, which may also be identified with a null bi-vector (see [18; p. 560]) on a bi-vector "à la E. Cartan". Although yet at a pre-historic stage, this corollary remark should be kept somewhere in mind for future developments.

Generators of the Lorentz transformations and trivial splits of extended exterior products

Proposition 03: the 4D source of a bi-vector

Any bi-vector has 6 complex components (see, e. g.: [13]); they always can be organized inside a skew-symmetric (4-4) matrix with a null diagonal. That matrix can always be thought as the trivial decomposition of some deformed tensor or Lie product involving some 4D complex vector.

Remark 06: the generators of the Lorentz transformations

The matrix representing the generators of the Lorentz transformations is a natural skew-symmetric (4-4) matrix with a null diagonal. It has the following generic formalism:
(57)

$$[\ldots M^{\mu\nu} \ldots] = \begin{bmatrix} 0 & <\mathbf{K}| \\ -|\mathbf{K}> & ^{(3)}[J] \end{bmatrix}$$

Where $^{(3)}\mathbf{K}$: ($K^1 = M^{01}$, $K^2 = M^{02}$, $K^3 = M^{03}$) represents the boost and [J] the rotation.

Remark 07: the 4D source of a bi-3Dvector

Let suppose that there exists an unknown 4D complex vector $^{(4)}\mathbf{X}$ which is element of ($E_4(C)$, $⌂_{\blacktriangledown A}$) where the symbol $⌂_{\blacktriangledown A}$ represents an extended "exterior" product (this means that the two indices of the cube ▼A have an anti-symmetric behavior; in extenso: $A_{\lambda\mu}{}^{\nu} + A_{\mu\lambda}{}^{\nu} = 0$). Any extended exterior product of the following type, $⌂_{\blacktriangledown A}$(^(4à**X**, …), has a trivial split which one can write $_{\blacktriangledown A}\Phi(^{(4)}\mathbf{X})$. If furthermore that trivial matrix is skew-symmetric, then: $_{\blacktriangledown A}\Phi(^{(4)}\mathbf{X}) + \,_{\blacktriangledown A}\Phi^t(^{(4)}\mathbf{X}) = 0$ (where the subscript " t " positioned behind/after a matrix denotes the transposed of that matrix) and the following rule holds true for the components of the cube:
(58)

$$A_{\lambda\mu}{}^{\varepsilon} = - A_{\lambda\varepsilon}{}^{\mu} = A_{\varepsilon\lambda}{}^{\mu} = - A_{\varepsilon\mu}{}^{\lambda} = A_{\mu\varepsilon}{}^{\lambda} = - A_{\mu\lambda}{}^{\varepsilon}.$$

It can be proven that that matrix has always the formalism of a matrix representing a bi-3Dvector (Annex 03).

Theorem 03: Existence of a 4D source for the generators of the Lorentz transformations

From the three previous remarks one induces the existence of some unknown 4D complex vector $^{(4)}\mathbf{X}$ and of some unknown anti-symmetric cube ▼A such that any representation of the generators of the Lorentz transformation can be written:
(59)

$$\exists \blacktriangledown A \mid (02) \text{ and } \exists \,^{(4)}\mathbf{X} \mid [\ldots M^{\mu\nu} \ldots] = \,_{\blacktriangledown A}\Phi(^{(4)}\mathbf{X})$$

Corollary 03: a first plausible formalism for the source

The relation (59) writes:
(60)

$$M^{\mu\nu} = A_{\alpha\nu}{}^{\mu} . X^{\alpha}$$

And this suggests that the formalism of the unknown source may perhaps be:

$$X_{\varepsilon} = ½. \sum_{\mu} \sum_{\nu} \,_{1}A_{\mu\nu}{}^{\varepsilon} . M^{\mu\nu} = ½. \sum_{j} \sum_{k} \,_{1}A_{jk}{}^{\varepsilon} . M^{jk} + ½. \sum_{j} \,_{1}A_{0k}{}^{\varepsilon} . M^{0k} + ½. \sum_{j} \sum_{k} \,_{1}A_{j0}{}^{\varepsilon} . M^{j0}$$

The double anti-symmetry constraint on ▼A and on [… $M^{\mu\nu}$ …] yields a reasonable proposition:
(61)

$$X_{\varepsilon} = ½. \sum_{j} \sum_{k} A_{jk}{}^{\varepsilon} . M^{jk} + \sum_{a} A_{0a}{}^{\varepsilon} . K^{a} \text{ where } \varepsilon = 0, 1, 2, 3 \text{ whilst j, k and a} = 1, 2,3$$

Corollary 04: systematic existence of a pre-metric

When the relations (60) and (61) are compatible:

(62)

$$X_{\varepsilon} = ½. \sum_{\mu} \sum_{\nu} A_{\mu\nu}{}^{\varepsilon} . M^{\mu\nu} = ½. \sum_{\mu} \sum_{\nu} A_{\mu\nu}{}^{\varepsilon} . A_{\alpha\nu}{}^{\mu} . X^{\alpha}$$

And there exists plausibly a mathematical tool insuring the lowering and the rising from the subscripts and the indices on $(E_4(C),$ ⌂▼$_A)$:

(63)

$$½. \sum_{\mu} \sum_{\nu} A_{\mu\nu}{}^{\varepsilon} . A_{\alpha\nu}{}^{\mu} = G_{\varepsilon\alpha}$$

This formalism is remembering the one introduced in [19; p. 139, (1.3)] which will be analyzed later. The notion of pre-metric has a long history in physics. More exactly one should recall here the courageous attempts to (re)build a generalized theory of relativity [20] or the Maxwell's laws without the pre-existence of a metric [21]. In all cases the sophisticated relations between vectors, bi-vectors and spinors play a determinant role.

Proposition 04: the kinetic momentum as the source of the generators of the Lorentz transformations?

I shall investigate the following and strange cases where the source of the generator of a Lorentz generator is an event in a phase space: ${}^{(4)}\mathbf{p}$. This will perhaps allow to interpret (63) as the local 4D metric. I am in fact proposing this hypothesis because of the condition validating the theorem 02 and because of the informations contained in the annex 03.

Conclusion

In this document, I have proved that, when the EM fields can be written as explained in [a], then they may sometimes be equivalent to small variations of the geometry. This is the reason why I have called that part of my explorations: "The subtle interplay between the EM fields and the geometry". This is not a new document (2016), but it contains interesting and pedagogical information about the links between EM fields and spinors.

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Annexes

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Annex 01: Some useful calculations for the future

Considering the definition of the Christoffel's symbols of the second kind [07; p. 18, (1.44)], it is easy to check that:

$$\Gamma_{\lambda\mu}{}^{\sigma}. g_{\sigma\alpha} = g^{\upsilon\sigma}. \{\partial_{\lambda} g_{\mu\upsilon} + \partial_{\mu} g_{\lambda\upsilon} - \partial_{\upsilon} g_{\lambda\mu}\}. g_{\sigma\alpha}$$

Since we are working on R (or C) which is an associative set for the multiplication:

$$\Gamma_{\lambda\mu}{}^{\sigma}. g_{\sigma\alpha} = g^{\upsilon\sigma}. g_{\sigma\alpha}. \{\partial_{\lambda} g_{\mu\upsilon} + \partial_{\mu} g_{\lambda\upsilon} - \partial_{\upsilon} g_{\lambda\mu}\} = \delta^{\upsilon}{}_{\alpha}. \{\partial_{\lambda} g_{\mu\upsilon} + \partial_{\mu} g_{\lambda\upsilon} - \partial_{\upsilon} g_{\lambda\mu}\} = \{\partial_{\lambda} g_{\mu\alpha} + \partial_{\mu} g_{\lambda\alpha} - \partial_{\alpha} g_{\lambda\mu}\}$$

Now let us define:

$$\omega_{\lambda\mu\alpha} = \Gamma_{\lambda\mu}{}^{\sigma}.\, g_{\sigma\alpha} = \{\partial_\lambda g_{\mu\alpha} + \partial_\mu g_{\lambda\alpha} - \partial_\alpha g_{\lambda\mu}\}$$

We state that:

$$\omega_{\mu\lambda\alpha} = \Gamma_{\mu\lambda}{}^{\sigma}.\, g_{\sigma\alpha} = \{\partial_\mu g_{\lambda\alpha} + \partial_\lambda g_{\mu\alpha} - \partial_\alpha g_{\mu\lambda}\}$$

$$\omega_{\lambda\alpha\mu} = \Gamma_{\lambda\alpha}{}^{\sigma}.\, g_{\sigma\mu} = \{\partial_\lambda g_{\alpha\mu} + \partial_\alpha g_{\lambda\mu} - \partial_\mu g_{\lambda\alpha}\}$$

$$\omega_{\alpha\mu\lambda} = \Gamma_{\alpha\mu}{}^{\sigma}.\, g_{\sigma\lambda} = \{\partial_\alpha g_{\mu\lambda} + \partial_\mu g_{\alpha\lambda} - \partial_\lambda g_{\alpha\mu}\}$$

Let us suppose that we work in a region with a symmetric metric; (10-1) holds true again and:

$$\omega_{\mu\lambda\alpha} = \omega_{\lambda\mu\alpha}$$

$$\omega_{\lambda\alpha\mu} + \omega_{\lambda\mu\alpha} = 2.\, \partial_\lambda g_{\alpha\mu}$$

$$\omega_{\alpha\mu\lambda} + \omega_{\lambda\mu\alpha} = 2.\, \partial_\mu g_{\alpha\lambda}$$

$$\omega_{\mu\lambda\alpha} + \omega_{\lambda\alpha\mu} + \omega_{\alpha\mu\lambda} = \partial_\lambda g_{\alpha\mu} + \partial_\alpha g_{\lambda\mu} + \partial_\mu g_{\alpha\lambda}$$

Annex 02: in case of doubt

The formalism of all off-diagonal terms and the fact that we are working on R (or C) have a lucky consequence on the calculations; the off-diagonal terms vanish for the following matrix:

(a02-01)

$$\tfrac{1}{2}.\, \{X_1.\, X_2 + X_2.\, X_1\}$$

$$=$$

$$\begin{bmatrix} x_i^1.x_j^{1\prime} + x_i^2.x_j^{2\prime} + x_j^1.x_i^{1\prime} + x_j^2.x_i^{2\prime} & & & \\ & x_i^{1\prime}.x_j^1 + x_i^2.x_j^{2\prime} + x_j^{1\prime}.x_i^1 + x_j^2.x_i^{2\prime} & & \\ & & x_i^1.x_j^{1\prime} + x_i^{2\prime}.x_j^2 + x_j^1.x_i^{1\prime} + x_j^{2\prime}.x_i^2 & \\ & & & x_i^{1\prime}.x_j^1 + x_i^{2\prime}.x_j^2 + x_j^{1\prime}.x_i^1 + x_j^{2\prime}.x_i^2 \end{bmatrix}$$

Since all elements of the diagonal are equal, this formalism is unfortunately not compatible with the one of η (13).

Annex 03: double constraint on the cubes and bi-vector

Proposition: If the components of an anti-symmetric cube are such that:

(A01-a)

$$A_{\lambda\mu}{}^{\varepsilon} = -\, A_{\lambda\varepsilon}{}^{\mu}$$

then the projection of any "projectile", say [(4)]**a**, is the representation of a bi-vector. Let us write that projection:

(A01-b)

$$\blacktriangledown_A \Phi(a) = \begin{bmatrix} A_{\gamma 0}{}^{0}.a^\gamma & A_{\gamma 1}{}^{0}.a^\gamma & A_{\gamma 2}{}^{0}.a^\gamma & A_{\gamma 3}{}^{0}.a^\gamma \\ A_{\gamma 0}{}^{1}.a^\gamma & A_{\gamma 1}{}^{1}.a^\gamma & A_{\gamma 2}{}^{1}.a^\gamma & A_{\gamma 3}{}^{1}.a^\gamma \\ A_{\gamma 0}{}^{2}.a^\gamma & A_{\gamma 1}{}^{2}.a^\gamma & A_{\gamma 2}{}^{2}.a^\gamma & A_{\gamma 3}{}^{2}.a^\gamma \\ A_{\gamma 0}{}^{3}.a^\gamma & A_{\gamma 1}{}^{3}.a^\gamma & A_{\gamma 2}{}^{3}.a^\gamma & A_{\gamma 3}{}^{3}.a^\gamma \end{bmatrix}$$

With an anti-symmetric cube and with (A01-a), it is easy to check that:

(A01-c)

$$\blacktriangledown_A \Phi(a) = \begin{bmatrix} 0 & A_{21}{}^{0}.a^2 + A_{31}{}^{0}.a^3 & A_{12}{}^{0}.a^1 + A_{32}{}^{0}.a^3 & A_{13}{}^{0}.a^1 + A_{23}{}^{0}.a^2 \\ A_{20}{}^{1}.a^2 + A_{30}{}^{1}.a^3 & 0 & A_{02}{}^{1}.a^0 + A_{32}{}^{1}.a^3 & A_{03}{}^{1}.a^0 + A_{23}{}^{1}.a^2 \\ A_{10}{}^{2}.a^1 + A_{30}{}^{2}.a^3 & A_{01}{}^{2}.a^0 + A_{31}{}^{2}.a^3 & 0 & A_{03}{}^{2}.a^0 + A_{13}{}^{2}.a^1 \\ A_{10}{}^{3}.a^1 + A_{20}{}^{3}.a^2 & A_{01}{}^{3}.a^0 + A_{21}{}^{3}.a^2 & A_{02}{}^{3}.a^0 + A_{12}{}^{3}.a^1 & 0 \end{bmatrix}$$

This can be simplified with:

(A01-d)

$$\blacktriangledown_A \Phi(a) = \begin{bmatrix} 0 & x^2 + x^3 & -x^1 + x^3 & -x^1 - x^2 \\ -x^2 - x^3 & 0 & x^0 + x^3 & x^0 - x^2 \\ x^1 - x^3 & -x^0 - x^3 & 0 & x^0 + x^1 \\ x^1 + x^2 & -x^0 + x^2 & -x^0 - x^1 & 0 \end{bmatrix}$$

If we conventionally write:
(A01-e)

$$X^1 = -(x^2 + x^3)$$

$$X^2 = -x^1 + x^3$$

$$X^3 = x^1 + x^2$$

$$Y^3 = -x^0 + x^3$$

$$Y^2 = x^0 - x^2$$

$$Y^1 = -x^0 + x^1$$

This is just:
(A01-f)

$$\blacktriangledown_A \Phi(a) = \begin{bmatrix} 0 & -X^1 & X^2 & -X^3 \\ X^1 & 0 & -Y^3 & Y^2 \\ -X^2 & Y^3 & 0 & -Y^1 \\ X^3 & -Y^2 & Y^1 & 0 \end{bmatrix}$$

which obviously represents a bi-3Dvector (X*, Y) with X* : (X^1, $-X^2$, X^3) et Y : (Y^1, Y^2, Y^3). □